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Sufficient Optimality Condition for a Risk-Sensitive Control Problem for Backward Stochastic Differential Equations and an Application

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Abstract. *We study the risk-sensitive optimal control by using the extending part of the same problem with a backward stochastic differential equation as has been reported in [2]. We also establish sufficient optimality conditions, by means of the convexity propriety of pertaining functions. The control domain is assumed to be convex, and the generator coefficient of the associated system is allowed to depend on the control variable. An example is provided to illustrate our main result for a risk-sensitive control problem under linear stochastic dynamics with an exponential quadratic cost function.*

Key words : Backward Stochastic Differential Equations, Risk-Sensitive, Sufficient Optimality Conditions, Variational Principle, Logarithmic Transformation.

AMS Subject Classifications : 93E20, 60H30, 60H10, 91B28

1. Introduction

In this paper, we investigate sufficient optimality conditions for the system driven by a backward stochastic differential equation in a risk sensitive model for the performance functional. In particular, we aim at a certain extension of an initial work, reported by Chala in [2], where a necessary optimality condition, of the Pontryagin's maximum principle type, for risk-sensitive performance functionals, had been established. In [2] the problem was in fact solved by using an approach developed by Djehiche et al. [1]. For more details, the interested reader is referred to the papers [2, 1], and to references therein.

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In a risk-sensitive control problem, the system is governed by the nonlinear backward stochastic differential equation

$$\begin{cases} dy_t^v = -f(t, y_t^v, z_t^v, v_t)dt + z_t^v dW_t, \\ y_T^v = a. \end{cases}$$

The criterion to be minimized, with an initial risk-sensitive functional cost, can be defined as follows

$$J^\theta(v) = \mathbb{E}\left[e^{\theta\left\{ \Psi(y_0^v) + \int_0^T f(t, y_t^v, z_t^v, v_t)dt \right\}} \right].$$

A control u is called optimal if it solves $J^\theta(u) = \inf_{v \in \mathcal{U}} J^\theta(v)$.

A stochastic maximum principle (SMP in short) for risk-sensitive optimal control problems for Markov diffusion processes, with an exponential integral performance functional was obtained in [6] by relating the SMP to the dynamic programming principle (DPP in short). Specifically, the authors of [6] have used the first order adjoint process as the gradient of the value function of the control problem. Such a relationship holds, however, only when the value function is smooth (see Assumption (B4) in [6]). Moreover, by utilizing the smoothness assumption, the two papers [8] and [9], have used the approach noted above, but extended to jump processes and to a linear system with an application to finance.

The existence of an optimal solution to the posing problem, of risk-sensitive control, is investigated in this paper, and some sufficient optimality conditions for the pertaining model are established. A few sentences are in order here about paper [2]. In this paper, we have reformulated the same problem in terms of an augmented state process and a terminal payoff problem. An intermediate stochastic maximum principle is then obtained by applying the SMP of [11], Theorem 3.1) for loss functionals without a running cost. Then, we transformed the intermediate first order adjoint processes to a simpler form by assuming convexity for the controls. Necessary optimality conditions were then established by using the logarithmic transform introduced in [3]. In this respect, the method of Lim and Zhou [6], shows in fact that it suffices to use a generic square-integral martingale to transform the pair (p_1, q_1) into the adjoint process $(\widetilde{p}_1(t), 0)$, where the process $\widetilde{p}_1(t)$ is still a square-integrable martingale, which would mean that $\widetilde{p}_1(t) = \widetilde{p}_1(T)$, and is equal to the constant $\mathbb{E}[\widetilde{p}_1(T)]$. But this generic martingale needs not be related to the adjoint process $\vec{p}(t)$, as in [6]. Instead, it will be part of the adjoint equation associated with the risk-sensitive SMP (see theorem 3.2, bellow, or Theorem 3.2, page 409 in [2]).

The rest of this paper is organized as follows. In Section 2, we give the precise formulation of the problem and introduce the risk-sensitive model, together with the various assumptions, as in [2], used throughout the paper. In section 3, we give only the important results and study our system of backward SDE. The necessary optimality condition for the backward differential equation, with risk-sensitive performance cost, is also given here. Our main result : the sufficient optimality conditions for the risk-sensitive control problem, under an additional hypothesis, is provided in section 4. In sections 5, we finish the paper by giving an application to a quadratic stochastic linear control problem. In the conclusion and outlook, of section 6, we give a discussion of future research challenges, and outline the relationship between this paper and other related works.

2. Formulation of the Problem

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\in[0,T]}, \mathbb{P}\right)$ be a probability space satisfying the usual conditions, in which a *one*–dimensional Brownian motion $W = (W_t : 0 \leq t \leq T)$ is defined. We assume that $(\mathcal{F}_t^W)_{t\in[0,T]}$ is defined by $\forall t \geq 0,\ \mathcal{F}_t^W = \sigma\left(W_r;\ \text{for any } r \in [0,t]\right) \vee \mathcal{N},$ where $\mathcal{N}$ denotes the totality of $\mathbb{P}$–null sets. Let $\mathcal{M}^2(0,T;\mathbb{R})$ denotes the set of *one*–dimensional jointly measurable random processes $\{\varphi_t, t \in [0,T]\},$ which satisfy:

$(i) : \mathbb{E}\left[\int_0^T |\varphi_t|^2 dt\right] < \infty,\ (ii) : \varphi_t$ is $(\mathcal{F}_t^W)_{t\in[0,T]}$ measurable, for any $t \in [0,T]$.

We denote similarly by $\mathcal{S}^2([0,T];\mathbb{R})$ the set of continuous *one* dimensional random processes which satisfy:

$(i) : \mathbb{E}\left[\sup_{0\leq t\leq T} |\varphi_t|^2\right] < \infty,\ (ii) : \varphi_t$ is $(\mathcal{F}_t^W)_{t\in[0,T]}$ measurable, for any $t \in [0,T]$.

We also let T be a strictly positive real number and U to be a nonempty subset of $\mathbb{R}$.

Definition 2.1. An admissible control v is a process with values in U such that $\mathbb{E}\left[\int_0^T |v_t|^2 dt\right] < \infty.$ We denote by $\mathcal{U}$ the set of all admissible controls.

The set of all admissible control should be convex. For any $v \in \mathcal{U},$ we consider the following backward stochastic differential equation system

$$\begin{cases} dy_t^v = -g(t, y_t^v, z_t^v, v_t)dt + z_t^v dW_t, \\ y_T^v = a, \end{cases} \tag{1}$$

where $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R},$ and the terminal condition $a \in \mathbb{R}$ is a random variable $\mathcal{F}_T$ -measurable.

We define the criterion to be minimized, with initial risk-sensitive performance cost, as follows

$$J^\theta(v) = \mathbb{E}\left[e^{\theta\left\{\Psi(y_0^v) + \int_0^T f(t, y_t^v, z_t^v, v_t)dt\right\}} \right], \tag{2}$$

where θ is the risk-sensitive index, and $\Psi : \mathbb{R} \to \mathbb{R},\ f : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}.$

The control problem is to minimize the functional J^θ over $\mathcal{U},$ if $u \in \mathcal{U}$ is an optimal control, that is

$$J^\theta(u) = \inf_{v\in\mathcal{U}} J^\theta(v). \tag{3}$$

Assumption 2.1. Assume that $g(t,0,0) \in \mathcal{M}^2(0,T;\mathbb{R}),$ and that there exists $c > 0,$ such that $|g(t,y_1,z_1) - (t,y_2,z_2)| \leq c(|y_1 - y_2| + \|z_1 - z_2\|).$

Proposition [7] **2.1**. *For any given admissible control* $v(.)$*, we let assumption 2.1 to hold. Then the backward SDE* (1) *has a unique solution.*

A control that solves the problem $\{(1),(2),(3)\}$ is called optimal, and that applies to the case where the necessary optimality conditions for risk-sensitive model as given in [2]. Our goal here is to establish the risk-sensitive sufficient conditions of optimality in the posing stochastic control problem. For that we need the assumption that follows.

Assumption 2.2. $g,\ f$ and Ψ are continuously differentiable with respect to (y,z,v).

Under the above assumption, for every $v \in \mathcal{U}$ equation (1) has a unique strong solution, and the cost function J^{θ} is well defined from $\mathcal{U}$ into $\mathbb{R}$. For more details, the reader can be referred to the famous paper [11], and also to Yong's book [12].

3. Risk-Sensitive Stochastic Maximum Principle of Backward Type Control

This section is basically taken from the paper [2]. We start by introducing an auxiliary state process ξ_t^v, which is solution of the following forward SDE:
$d\xi_t^v = f(t,y_t^v,z_t^v,v_t)dt, \quad \xi_0^v = 0.$

The backward type control problem $\{(1),(2),(3)\}$ is equivalent to

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[e^{\theta\{\Psi(y_0^v)+\xi_T\}}\right], \\ \text{subject to} \\ d\xi_t^v = f(t,y_t^v,z_t^v,v_t)dt, \\ dy_t^v = -g(t,y_t^v,z_t^v,v_t)dt + z_t^v dW_t, \\ \xi_0^v = 0, \qquad y_T^v = a. \end{cases} \tag{4}$$

Recall that

$$A_T^{\theta} := e^{\theta\left\{\Psi(y_0^u)+\int_0^T f(t,y_t^u,z_t^u,u_t)dt\right\}}. \tag{5}$$

We can put $\Theta_T = \Psi(y_0^u) + \int_0^T f(t,y_t^u,z_t^u,u_t)dt,$ the risk-sensitive loss functional is given by

$$\begin{aligned} \Theta_\theta &=: \frac{1}{\theta}\log\mathbb{E}\left(\exp\theta\left\{\Psi(y_0^u) + \int_0^T f(t,y_t^u,z_t^u,u_t)dt\right\}\right) \\ &= \frac{1}{\theta}\log(\mathbb{E}\{\exp\theta\Theta_T\}). \end{aligned}$$

When the risk-sensitive index θ is small, the loss functional Θ_θ can be expanded as

$$\mathbb{E}(\Theta_T) + \tfrac{\theta}{2}Var(\Theta_T) + O(\theta^2),$$

where, $Var(\Theta_T)$ denotes the variance of Θ_T. If $\theta < 0,$ the variance of $\Theta_T,$ as a measure of risk, improves the performance $\Theta_\theta,$ in which case the optimizer is called *risk seeker*. But, when $\theta > 0,$ the variance of Θ_T worsens the performance Θ_θ, in which case the optimizer is called *risk averse*. The risk-neutral loss functional $\mathbb{E}(\Theta_T)$ can be conceived as a limit of risk-sensitive functional Θ_θ when $\theta \to 0$. For more details, the reader can consult the papers [1, 3, 10].

Notation 3.1. The following notations shall be used throughout the paper. For every $\phi = g,\ f$

respectively, we define

$$\begin{cases} \phi(t) = \phi(t,y_t^u,z_t^u,u_t), \\ \partial\phi(t) = \phi(t,y_t^u,z_t^u,v_t) - \phi(t,y_t^u,z_t^u,u_t), \\ \phi_\zeta(t) = \frac{\partial\phi}{\partial\zeta}(t), \qquad \forall\zeta = y,\ z, \end{cases}$$

where v_t in an admissible control from $\mathcal{U}$.

If we assume that assumptions 2.1-2.2 hold, then the adjoint equation can be found by using the stochastic maximum principle for risk-neutral of forward-backward type control, [11], with augmented state dynamics (ξ,y,z). There exist unique $\mathcal{F}_t$ –adapted pairs of processes $((p_1,q_1),(p_2,q_2))$, which solve the following system of the compact-backward stochastic differential equation:

$$\begin{cases} d\vec{p}(t) = \begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}dt \\ \qquad\qquad\qquad\qquad + \begin{pmatrix} q_1(t) \\ -\widetilde{H}_z^\theta(t) \end{pmatrix}dW_t, \\ p_1(T) = -\theta A_T^\theta, \qquad\qquad p_2(0) = -\theta\Psi_y(y_0^u)A_T^\theta, \end{cases} \tag{6}$$

with

$$\mathbb{E}\left[\sum_{i=1}^{2}\sup_{0\le t\le T}|p_i(t)|^2 + \int_0^T|q_1(t)|^2dt\right] < \infty.$$

We suppose here that $\widetilde{H}^\theta$ is the neutral Hamiltonian associated with the optimal state dynamics (ξ,y,z), and that the pair of adjoint process $\left(\vec{p}(t),\vec{q}(t)\right)$ $\left(6\right)$, is given by:

$$\widetilde{H}^\theta\left(t,y_t^u,z_t^u,u_t,\vec{p}(t)\right) := f(t)p_1(t) + g(t)p_2(t).$$

The following theorem is called the stochastic maximum principle for risk-neutral forward-backward type control from.

Theorem 3.1. *Assume that assumptions 2.1-2.2 hold. If* (ξ,y^u,z^u) *is an optimal solution of the risk-neutral control problem* $\left(4\right)$, *then there are two pairs of* F_t *–adapted processes* (p_1,q_1), *and* (p_2,q_2) *that satisfy* $\left(6\right)$, *such that*

$$\partial\widetilde{H}^\theta(t) \le 0, \tag{7}$$

for all $u \in U$, *almost every* $t \in [0,T]$, *and* P *–almost surely, where*

$$\partial\widetilde{H}^\theta(t) := \widetilde{H}^\theta\left(t,\xi_t^u,y_t^u,z_t^u,v,\vec{p}(t)\right) - \widetilde{H}^\theta\left(t,\xi_t^u,y_t^u,z_t^u,u_t,\vec{p}(t)\right).$$

Proof. The proof of (7) can be found in [11]. ■

In this section, we will state, without proof, the necessary conditions of optimality for the system driven by a backward stochastic differential equation with a risk sensitive performance type. For this end, let us summarize some of lemmas that are needed later.

Lemma [2] **3.1**. *V^θ solves the following linear backward SDE*

$$dV^\theta(t) = \theta l(t) V^\theta(t) dW_t, \quad V^\theta(T) = A_T^\theta. \tag{8}$$

Hence, the process defined on $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^{(W)}\right)_{t\in[0,T]}, \mathbb{P}\right)$ by L_t^θ, where

$$\frac{V^\theta(t)}{V^\theta(0)} = \exp\left(\int_0^t \theta l(s) dW_s - \frac{\theta^2}{2}\int_0^t |l(s)|^2 ds\right) := L_t^\theta, \quad 0 \le t \le T. \tag{9}$$

is a uniformly bounded F_t –martingale.

Furthermore, we may define the adjoint equation adopted to this kind of problem by invoking the next lemma.

Lemma [2] **3.2**. *The risk-sensitive adjoint equation satisfied by $(\widetilde{p}_2, \widetilde{q}_2)$ and (V^θ, l) becomes*

$$\begin{cases} d\widetilde{p}_2(t) = -H_y^\theta(t)dt - H_z^\theta(t)dW_t^\theta, \\ dV^\theta(t) = \theta l(t) V^\theta(t) dW_t, \\ V^\theta(T) = A_T^\theta, \\ \widetilde{p}_2(0) = -\Psi_y(y_0). \end{cases} \tag{10}$$

The solution $(\widetilde{p}, \widetilde{q}, V^\theta, l)$ of the system (10) is unique, such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |\widetilde{p}(t)|^2 + \sup_{0\le t\le T} |V^\theta(t)|^2 + \int_0^T \left(|\widetilde{q}(t)|^2 + |l(t)|^2\right) dt\right] < \infty, \tag{11}$$

where

$$H^\theta\left(t, y_t, z_t, \begin{pmatrix} \widetilde{p}_2 \\ \widetilde{q}_2 \end{pmatrix}, V^\theta, l\right) = (g(t) + z\theta l(t))\widetilde{p}_2(t) - f(t). \tag{12}$$

Theorem 3.2. (Risk-sensitive stochastic maximum principle) *Assume that assumption 2.1 holds. If (y_t, z_t, u_t) is an optimal solution of the risk-sensitive control problem $\{(1), (2), (3)\}$, then there exist two pairs of F_t-adapted processes (V^θ, l), (p, q) that satisfy $(10) - (11)$ such that*

$$\partial H^\theta(t) \le 0, \tag{13}$$

for all $u \in U$, almost every $0 \le t \le T$ and P-almost surely, where the Hamiltonian H^θ associated with (4), is given by

$$\widetilde{H}^\theta\left(t, \xi_t^u, y_t^u, z_t^u, \vec{p}(t), u_t\right) = \{\theta V_t^\theta\} H^\theta\left(t, y_t^u, z_t^u, \begin{pmatrix} \widetilde{p}_2(t) \\ \widetilde{q}_2(t) \end{pmatrix}, V_t^\theta, l(t), u_t\right),$$

and H^θ is the risk-sensitive Hamiltonian given by (12).

4. Sufficient Optimality Conditions for a Risk-Sensitive Performance Cost

In this section, we study when the necessary optimality conditions (13) become sufficient. For any $v \in \mathcal{U}$, we denote by (y_t^v, z_t^v) the solution of equation (1) controlled by v, to state the result that follows.

Theorem 4.1. (Sufficient optimality conditions) *Assume that the functions* Ψ, *and* $(y_t^v, z_t^v) \mapsto H^\theta(t, y_t^v, z_t^v, \widetilde{p}_1(t), \widetilde{p}_2(t), V_t^\theta, l(t), v(t))$ *are convex, and that for any* $v \in U$, $y_T^v = a$ *is an one –dimensional* F_T *–measurable random variable such that* $E|a|^2 < \infty$. *Then, u is an optimal solution of the control problem* $\{(1), (2), (3)\}$, *if it satisfies* (13).

Proof. Let u_2 be an arbitrary element of $\mathcal{U}$ (candidate to be optimal). For any $u_1 \in \mathcal{U}$, we have

$$\begin{aligned} & J^\theta(u_1(t)) - J^\theta(u_2(t)) \\ & = \mathbb{E}\left[e^{\theta\left\{\Psi\left(y_0^{u_1}\right) + \xi_T^{u_1}\right\}} \right] - \mathbb{E}\left[e^{\theta\left\{\Psi\left(y_0^{u_2}\right) + \xi_T^{u_2}\right\}} \right] \\ & = \mathbb{E}\left[\theta e^{\theta\left\{\Psi\left(y_0^{u_2}\right) + \xi_T^{u_2}\right\}} \{(\xi_T^{u_1} - \xi_T^{u_2}) + \Psi_y(y_0^{u_2})((y_0^{u_1}) - (y_0^{u_2}))\} \right]. \end{aligned}$$

Because Ψ is convex, we can write

$$\begin{aligned} & J^\theta(u_1(t)) - J^\theta(u_2(t)) \\ & \geq \mathbb{E}\left[\theta e^{\theta\left\{\Psi\left(y_0^{u_2}\right) + \xi_T^{u_2}\right\}} (\xi_T^{u_1} - \xi_T^{u_2}) \right] \\ & + \mathbb{E}\left[\theta e^{\theta\left\{\Psi\left(y_0^{u_2}\right) + \xi_T^{u_2}\right\}} \Psi_y(y_0^{u_2})((y_0^{u_1}) - (y_0^{u_2})) \right]. \end{aligned}$$

It follows from (6), and (5), that $p_1(T) = \theta A_T^\theta, p_2(0) = \theta \Psi_y(y_0^{u_2}) A_T^\theta$, and then we have

$$J^\theta(u_1(t)) - J^\theta(u_2(t)) \geq \mathbb{E}[p_1(T)(\xi_T^{u_1} - \xi_T^{u_2})] + \mathbb{E}[p_2(0)((y_0^{u_1}) - (y_0^{u_2}))]. \tag{14}$$

Applying Itô's formula to $p_1(t)(\xi_t^{u_1} - \xi_t^{u_2})$, and $p_2(t)((y_t^{u_1}) - (y_t^{u_2}))$ lead to

$$\mathbb{E}[p_1(T)(\xi_T^{u_1} - \xi_T^{u_2})] = \mathbb{E}\int_0^T p_1(t)(f^{u_1}(t) - f^{u_2}(t))dt,$$

and

$$\begin{aligned} \mathbb{E}[p_2(0)((y_0^{u_1}) - (y_0^{u_2}))] = & -\mathbb{E}\int_0^T (f_y^{u_2}(t)p_1(t) + g_y^{u_2}(t)p_2(t))(y_t^{u_1} - y_t^{u_2})dt \\ & - \mathbb{E}\int_0^T (f_z^{u_2}(t)p_1(t) + g_z^{u_2}(t)p_2(t))(z_t^{u_1} - z_t^{u_2})dt \\ & + \mathbb{E}\int_0^T p_1((g^{u_1}(t) - g^{u_2}(t)))dt. \end{aligned}$$

By substituting the above two formulas in (14), we get

$$
\begin{aligned}
&J^{\theta}(u_1(t)) - J^{\theta}(u_2(t)) \\
&\geq \mathbb{E}\int_0^T \Big[\widetilde{H}^{\theta}(t, y_t^{u_1}, z_t^{u_1}, p_1(t), p_2(t), u_1(t)) \\
&- \widetilde{H}^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t)) \Big] dt \\
&- \mathbb{E}\int_t^T \widetilde{H}_y^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(y_t^{u_1} - y_t^{u_2}) dt \\
&- \mathbb{E}\int_t^T \widetilde{H}_z^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(z_t^{u_1} - z_t^{u_2}) dt.
\end{aligned}
\tag{15}
$$

Since H is convex with respect to (y_t, z_t, u_t), then by considering the gradient of H, evaluated at $(y_t^{u_2}, z_t^{u_2}, v)$, and the necessary optimality conditions (13) and (15), we arrive at

$$
\begin{aligned}
&\Big[\widetilde{H}^{\theta}(t, y_t^{u_1}, z_t^{u_1}, p_1(t), p_2(t), u_1(t)) \\
&- \widetilde{H}^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t)) \Big] \\
&\geq \widetilde{H}_y^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(y_t^{u_1} - y_t^{u_2}) \\
&+ \widetilde{H}_z^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(z_t^{u_1} - z_t^{u_2}) \\
&+ \widetilde{H}_u^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(u_1(t) - u_2(t)),
\end{aligned}
$$

or equivalently at

$$
\begin{aligned}
&\mathbb{E}\int_0^T \widetilde{H}_u^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(u_1(t) - u_2(t)) dt \\
&\geq \mathbb{E}\int_0^T \Big[\widetilde{H}^{\theta}(t, y_t^{u_1}, z_t^{u_1}, p_1(t), p_2(t), u_1(t)) \\
&- \widetilde{H}^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t)) \Big] dt \\
&- \mathbb{E}\int_0^T \widetilde{H}_y^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(y_t^{u_1} - y_t^{u_2}) dt \\
&- \mathbb{E}\int_0^T \widetilde{H}_z^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(z_t^{u_1} - z_t^{u_2}) dt.
\end{aligned}
$$

Then from (15), and also according to the necessary optimality conditions (7), we observe that

$$
J^{\theta}(u_1(t)) - J^{\theta}(u_2(t)) \geq \mathbb{E}\int_0^T \widetilde{H}_u^{\theta}(t, y_t^{u_2}, z_t^{u_2}, p_1(t), p_2(t), u_2(t))(u_1(t) - u_2(t)) dt \geq 0.
$$

Which means that $J^{\theta}(u_1(t)) - J^{\theta}(u_2(t)) \geq 0.$ Here the proof completes. ■

5. Application to the Quadratic Risk-Sensitive Linear Control Problem

The basic securities consist of two assets; one of them is risky means that the money market is bond with price P^0 governed by the equation

$$dP_t^0 = P_t^0 r_t dt,$$

where r_t is the short rate. The price process P for the stock is modeled by the linear stochastic differential equation

$$dP_t = P_t[b_t dt + \sigma_t dW_t],$$

where W is a standard Brownian motion on $\mathbb{R},$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}).$ We further assume that the rate r is a bounded nonnegative process , the scalar of the stock rate b is a bounded process. σ is called the volatility, a bounded process, and λ is called risk premium. Here we note that $b_t - r_t\chi = \lambda_t\sigma_t,$ where χ is the scalar whose every component is 01.

Let us consider an investor whose actions cannot affect the market prices and who can decide at time $t \in [0,T]$ what amount u_t of the wealth V_t to invest in this stock. Here, we say a strategy is self-financing. The starting price of the claim is an initial endowment to guarantee $a.$ Following El Karoui et al. [4], a hedging strategy against a is feasible self-financing strategy (y,u) such that $y_T = a,$ where a is a positive square-integrable contingent claim, then our hedging strategy (y,u) against $a,$ such that

$$dy_t = dy_t^v = (ry_t^v + v_t\sigma\lambda)dt + v_t\sigma dW_t,$$

is such that the market value y is the fair price and the upper price of the claim.

The investor wants to minimize the functional cost of a risk-sensitive type of the following model

$$\mathbb{E}\left[\exp\theta\left\{\tfrac{1}{2}\int_0^T (v_t^2 + y_t^2)dt + \tfrac{1}{2}y_0^2\right\}\right].$$

Next, we provide a concrete example of risk-sensitive backward stochastic linear quadratic (LQ) problem, and give the explicit optimal portfolio and validate our major theoretical results in theorem 4.1 (Risk-sensitive sufficient optimality conditions). Consider the following quadratic risk-sensitive linear control problem

$$\begin{cases} \inf_{v \in \mathcal{U}} \mathbb{E}\left[\exp\theta\left\{\tfrac{1}{2}\int_0^T (v_t^2 + y_t^2)dt + \tfrac{1}{2}y_0^2\right\}\right], \\ \textit{subject to :} \\ dy_t^v = (ry_t^v + v_t\sigma\lambda)dt + v_t\sigma dW_t, \\ y_T^v = a. \end{cases} \tag{16}$$

Recall from (5), that $A_T := \exp\theta\left\{\frac{1}{2}\int_0^T (v_t^2 + y_t^2)dt + \frac{1}{2}y_0^2\right\}.$ Instantly, we give the Hamiltonian H^θ defined by

$H^\theta(t, y_t, z_t, v_t, \widetilde{p}_2(t), l(t)) = (ry_t + v_t\lambda\sigma + z_t\theta l(t))\widetilde{p}_2(t) - \frac{1}{2}(v_t^2 + y_t^2).$

Clearly $H_u^\theta(t, y_t, z_t, v_t, \widetilde{p}_2(t), l(t)) = \lambda\sigma\widetilde{p}_2(t) - v_t,$ and maximizing the Hamiltonian yields

$$u_t = \lambda\sigma\widetilde{p}_2(t). \tag{17}$$

Then, the optimal state dynamics is given by

$$\begin{cases} dy_t^u = (ry_t + \lambda^2\sigma^2\widetilde{p}_2(t))dt + v_t\sigma dW_t, \\ y_T^u = a. \end{cases} \tag{18}$$

Let y_t^u be a solution of (18) associated with optimal portfolio u_t. Then, there exists a unique adapted process $\widetilde{p}_2(t)$ of the following SDE system (called adjoint equation), according to equation (10). Let us define this equations as

$$\begin{aligned} d\widetilde{p}_2(t) &= -H_y^\theta(t)dt - H_z^\theta(t)dW_t^\theta, \\ \widetilde{p}_2(0) &= y_0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} H_y^\theta &= r\widetilde{p}_2(t) - y_t, \\ H_z^\theta &= \theta l(t)\widetilde{p}_2(t), \\ dW_t^\theta &= -\theta l(t)dt + dW_t, \\ A_T^\theta &= \exp\theta\left\{\tfrac{1}{2}\int_0^T (v_t^2 + y_t^2)dt + \tfrac{1}{2}y_0^2\right\}. \end{aligned} \tag{20}$$

At this point, we need only to prove that u is an optimal portfolio.

Theorem 5.1. *Suppose that the portfolio u satisfies* (17), *where $\widetilde{p}_2(t)$ satisfies* (19). *Then u is the unique optimal portfolio of the above backward stochastic differential equation of quadratic linear control problem* (16).

Proof. From the definition of the functional cost J^θ, we have

$$\begin{aligned} J^\theta(u) - J^\theta(v) &= \mathbb{E}\left[\exp\theta\left\{\tfrac{1}{2}\int_0^T (u^2 + y_t^u)dt + \tfrac{1}{2}y_0^2\right\}\right] \\ &\quad - \mathbb{E}\left[\exp\theta\left\{\tfrac{1}{2}\int_0^T (v_t^2 + y_t^v)dt + \tfrac{1}{2}y_0^2\right\}\right]. \end{aligned}$$

By applying Ito's formula, using the explicit forms of the adjoint equation (19), and following the same steps of the proof as in theorem 4.1, we can obtain that $J^\theta(u) - J^\theta(v) \geq 0$. This means that the portfolio u is the optimal process to our system of quadratic linear control with a the risk-sensitive performance functional.

The system governed by the equations (18) and (19) is a fully coupled forward backward stochastic differential equation. It is very hard to find the explicit solution to the system (18) and (19). To this end we must follow the next method. For any smooth deterministic functions $\alpha(t)$, $\beta(t)$, we can write the solution of $\widetilde{p}_2(t)$ as

$$\widetilde{p}_2(t) = \alpha(t)y_t + \beta(t). \tag{21}$$

By applying the Itô's formula to (21), we find that

$$\begin{aligned} d(\widetilde{p}_2(t)) &= \dot{\alpha}(t)y_t dt + \alpha(t)d(y_t) + \dot{\beta}(t)dt \\ &= \left[\left(\dot{\alpha}(t) + \alpha^2(t)\lambda^2\sigma^2 + \alpha(t)r\right)y_t + \alpha(t)\beta(t)\lambda^2\sigma^2 + \dot{\beta}(t)\right]dt \\ &\quad + \alpha(t)u_t\sigma dW_t. \end{aligned} \tag{22}$$

Furthermore, the adjoint equation (19), can be rewritten as follows

$$\begin{cases} d\widetilde{p}_2(t) = -(\alpha(t)ry_t + r\beta(t) - y_t)dt - \theta l(t)\widetilde{p}_2(t)dW_t^\theta, \\ \widetilde{p}_2(0) = y_0, \end{cases}$$

According to the proof of Lemma 3.2 in [2], it is very important to observe that

$dW_t^\theta = -\theta l(t)dt + dW_t,$ and $\widetilde{p}_2(t) = \alpha(t)y_t + \beta(t),$ as in $(21).$ Then the above adjoint equation implies that

$$\begin{cases} d\widetilde{p}_2(t) = [\{(1+\theta^2 l^2(t))\alpha(t)+1\}y_t + \theta^2 l^2(t)\beta(t)]dt \\ \qquad\qquad + \theta l(t)(\alpha(t)y_t + \beta(t))dW_t, \\ \widetilde{p}_2(0) = y_0, \end{cases} \tag{23}$$

By identifying (22) with $(23),$ we get

$$\begin{cases} \left(\dot{\alpha}(t) + \lambda^2\sigma^2\alpha^2(t) + r\alpha(t)\right) = (-r+\theta^2 l^2(t))\alpha(t) + 1 \\ \alpha(t)\beta(t)\lambda^2\sigma^2 + \dot{\beta}(t) = (\theta^2 l^2(t))\beta(t). \end{cases}$$

Then we can deduce that the above equation is the Riccati equation to $\alpha(t),$ and is given by

$$\begin{cases} \dot{\alpha}(t) + \lambda^2\sigma^2\alpha^2(t) + (2r - \theta^2 l^2(t))\alpha(t) + 1 = 0, \\ \dot{\alpha}(T) = 0. \end{cases} \tag{24}$$

The second step in identifying the coefficients, would be to list the ordinary differential equation

$$\begin{cases} \dot{\beta}(t) + (\lambda^2\sigma^2\alpha(t) - \theta^2 l^2(t))\beta(t) = 0, \\ \beta(T) = 0. \end{cases} \tag{25}$$

The optimal portfolio (17) can be written as

$$u_t = \alpha(t)\lambda\sigma y_t + \lambda\sigma\beta(t), \tag{26}$$

where the deterministic functions $\alpha(t),$ and $\beta(t)$ have the solution given by $(24),$ and (25) respectively.

Theorem 5.2. *Assume that the pair* $(\alpha(t), \beta(t))$ *are the unique solution to* (24) *and* $(25).$ *Then the optimal portfolio of the problem* (16) *has the state feedback form* $(26).$

6. Conclusion and Outlook

This paper reports on one main result, theorem 4.1, which establishes the sufficient optimality conditions for backward stochastic differential equation system of risk sensitive performance. A result obtained using an almost similar scheme as in Khallout and Chala [5]. In this paper, a detailed proof can be found of the explicit solution to the Riccati equation and to the ordinary differential equation. The last paper can be considered as an extension of the backward differential equation into fully coupled forward backward SDE. The main tools in the proof are tightness and use of the risk neutral maximum principle (theorem 3.1), in addition to using the result of El Karoui et al. [3]. The present work relies, moreover, on the paper of Chala [2]. Its proof is based on the convexity conditions of the Hamiltonian function, and the initial term of the performance function. It should be noted that the risk sensitive control problems studied by Lim and Zhou in [6] are different from ours. Our results can be compared with maximum principle obtained by Khallout et al. [5], whose results shall be discussed in future new joint paper. In this paper we will generalize the last result to the fully coupled forward backward stochastic differential equation, which is motivated by an optimal portfolio choice problem in the financial market. Specifically, the motivation is by the model of control cash flow of a firm or project. For example, one can set the model of pricing and management of an insurance contract, a counterpart without mean field term, as in [1]. A problem to be thoroughly addressed in our future paper, where the dynamical system is governed by a fully coupled stochastic differential equation of mean field type. Remarkably, the maximum principle of risk-neutral control obtained by Shi and Wu [8] is quite similar to our theorem 3.1, but their adjoint equation and maximum conditions heavily depend on the risk sensitive parameter.

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