Mean-field Reflected Backward Doubly Stochastic DE With Continuous Coefficients*

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Abstract. We study the existence and uniqueness of the solutions to mean-field reflected backward doubly stochastic differential equation (MF-RBDSDE), when the driver \(f\) is Lipschitzian. We also study the existence in the case where the driver is of linear growth and continuous. In this case we establish a comparison theorem.

Key words: Backward Doubly SDE, Mean-field, Continuous Coefficients, Comparison Theorem.

AMS Subject Classifications: 60H10, 60H05

1. Introduction

After the earlier work of Pardoux & Peng (1990), the theory of backward stochastic differential equations (BSDEs in short) has a significant headway thanks to the many application areas. Several authors contributed in weakening the Lipschitzian assumption required on the drift of the equation (see Lepaltier & San Martin (1996), Kobylanski (1997), Mao (1995), Bahlali (2000)).

A new kind of backward stochastic differential equations was introduced by Pardoux & Peng [5] (1994),

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \]

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral \(dW_t\) and a backward stochastic integral \(dB_t\). They have proved the existence and uniqueness of solutions for BDSDEs under uniformly Lipschitzian conditions. Shi et al. [6] (2005) provided a comparison theorem which is very important in studying viscosity solution of SPDEs with stochastic tools.

Bahlali et al. [2] (2009) proved the existence and uniqueness of the solution to the following

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Reflected backward doubly stochastic differential equations (RBDSDEs) with one continuous barrier and uniformly Lipschitzian coefficients:

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T g(s, Y_s, Z_s) \, dB_s
+ K_T - K_t - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T.
\]

In a recent work of Buckdahn et al. [3] (2009), a notion of mean-field backward stochastic differential equation (MF-BSDEs in short) of the form

\[
Y_t = \xi + \int_t^T E\{f(s, \omega', \omega, Y_s(\omega), Y_s(\omega'), Z_s)\} \, ds - \int_t^T Z_s \, dW_s,
\]

with \( t \in [0, T] \), was introduced. The authors deepened the investigation of such mean-field BSDEs by studying them in a more general framework, with a general driver. They established the existence and uniqueness of the solution under uniformly Lipschitzian conditions. The theory of mean-field BSDE has been developed by several authors. Du et al. [4] (2001); established a comparison theorem and existence in the case linear growth and continuous condition. Shi et al. [6]; introduced and studied mean-field backward stochastic Volterra integral equations.

Mean-field Backward doubly stochastic differential equations

\[
Y_t = \xi + \int_t^T E\{f(s, \omega', \omega, Y_s(\omega), Y_s(\omega'), Z_s)\} \, ds
+ \int_t^T E\{g(s, \omega', \omega, Y_s(\omega'), Z_s)\} \, dB_s - \int_t^T Z_s \, dW_s,
\]

with \( t \in [0, T] \), are deduced by Ruimin Xu [7] (2012), who obtained the existence and uniqueness result of the solution with uniformly Lipschitzian coefficients and presented the connection between McKean-Vlasov SPDEs and mean-field BDSDEs.

In this paper, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real valued mean-field reflected backward doubly stochastic differential equation: with \( t \in [0, T] \)

\[
Y_t = \xi + \int_t^T E\{f(s, \omega, \omega', Y_s, Y_s', Z_s, Z_s')\} \, ds
+ \int_t^T E\{g(s, \omega, \omega', Y_s, Y_s', Z_s, Z_s')\} \, dB_s + K_T - K_t - \int_t^T Z_s \, dW_s.
\]  \( (1) \)

We establish the existence and uniqueness of solutions for equation (1) under uniformly Lipschitz conditions on the coefficients. In the case where the coefficient \( f \) is only continuous, we establish the existence of maximal and minimal solutions. In the case where the coefficient \( f \) is continuous with linear growth, we approximate \( f \) by a sequence of Lipschitz functions \( (f_n) \) and use a comparison theorem established here for MF-RBDSDEs.

The paper is organized as follows: In Sections 2, we give some notations, assumptions, and we define a solution of RBDSDE. In Section 3, we state our main results for existence and uniqueness in the case where the coefficients are Lipschitzian, and we present a comparison theorem. The case where the generator is continuous and linear growth is treated in section 4.
2. Notation, Assumptions and Definitions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $T > 0$. Let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively. For $t \in [0, T]$, we put,

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B,$$

and $\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_t^B = \sigma(B_s - B_t; t \leq s \leq T)$, completed with $P$-null sets. It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub $\sigma$-fields, and hence it is not a filtration. However $(\mathcal{G}_t)$ is a filtration.

Let $M^2_T(0, T, \mathbb{R}^d)$ denote the set of $d$–dimensional, jointly measurable stochastic processes $\{\varphi_t; t \in [0, T]\}$, which satisfy:

(a) $E\int_0^T |\varphi_t|^2 dt < \infty$.

(b) $\varphi_t$ is $\mathcal{F}_t$–measurable, for any $t \in [0, T]$.

We denote by $S^2_T([0, T], \mathbb{R})$, the set of continuous stochastic processes $\varphi_t$, which satisfy:

(a’) $E(\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty$.

(b’) For every $t \in [0, T]$, $\varphi_t$ is $\mathcal{F}_t$–measurable.

Let $(\Omega, \mathcal{F}, P) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$ be the (non-completed) product of $(\Omega, \mathcal{F}, P)$ with itself. We denote the filtration of this product space by $\mathcal{F} = \{\mathcal{F}_t = \mathcal{F}_t \otimes \mathcal{F}_t, 0 \leq t \leq T\}$. A random variable $\xi \in L^0(\Omega, \mathcal{F}, P)$ originally defined on $\Omega$ is extended canonically to $\tilde{\Omega} : \xi'(\omega', \omega) = \xi(\omega'), (\omega', \omega) \in \tilde{\Omega} = \Omega \times \Omega$. For every $\theta \in L^1(\tilde{\Omega}, \mathcal{F}, P)$, the variable $\theta(\cdot, \cdot) : \tilde{\Omega} \rightarrow \mathbb{R}$ belongs to $L^1(\tilde{\Omega}, \mathcal{F}, P)$, $P(d\omega) - a.s.$ We denote its expectation by

$$E'[\theta(\cdot, \cdot)] = \int_{\Omega} \theta(\omega', \omega)P(d\omega').$$

Notice that $E'[\theta] = E'[\theta(\cdot, \cdot)] \in L^1(\tilde{\Omega}, \mathcal{F}, P)$, and

$$E[\theta] = \left(\int_{\Omega} \theta dP\right) = \int_{\Omega} E'[\theta(\cdot, \cdot)]P(d\omega) = E[E'[\theta]].$$

Then we consider the following assumptions,

H1) Let $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be two measurable functions such that for every $(y, z, y', z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d, f(., y, z, y', z')$ and, $g(., y, z, y', z')$ belongs in $M^2(0, T, \mathbb{R})$

H2) There exist constants $L > 0$ and $0 < \alpha < \frac{1}{2}$, such that for every $(t, \omega) \in \Omega \times [0, T]$ and $(y, z, y', z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y_1, z_1, y'_1, z'_1) - f(t, y_2, z_2, y'_2, z'_2)| \leq L(|y_1 - y_2| + |y'_1 - y'_2| + |z_1 - z_2| + |z'_1 - z'_2|)$$

$$\times |g(t, y_1, z_1, y'_1, z'_1) - g(t, y_2, z_2, y'_2, z'_2)|^2 \leq L\left(|y_1 - y_2|^2 + |y'_1 - y'_2|^2\right) + \alpha \left(|y_1 - y_2|^2 + |z_1 - z_2|^2 + |z'_1 - z'_2|^2\right).$$

H3) Let $\xi$ be a square integrable random variable which is $\mathcal{F}_T$–measurable.
The obstacle \( \{S_t, 0 \leq t \leq T\} \), is a continuous \( \mathcal{F}_t \)-progressively measurable real-valued process satisfying \( E\left( \sup_{0 \leq t \leq T} (S_t)^2 \right) < \infty \).

We assume also that \( S_T \leq \zeta \) a.s.

**Definition 2.1.** A solution of equation (1) is a \((\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+)\) -valued \( \mathcal{F}_t \)-progressively measurable process \((Y_t, Z_t, K_t)_{0 \leq t \leq T}\) which satisfies equation (1) and

i) \((Y_t, Z_t, K_t) \in S^2 \times M^2 \times L^2(\Omega)\).

ii) \( Y_t \geq S_t \).

iii) \((K_t)\) is continuous and nondecreasing, \( K_0 = 0 \) and \( \int_0^T (Y_t - S_t) dK_t = 0 \).

**3. Existence of a Solution to the RBDSDE With a Lipschitz Condition**

**Theorem 3.1.** Under conditions, H1), H2), H3) and H4), the MF-RBDSDE (1) has a unique solution.

**Proof.** For any \((y, z)\) we consider the following MF-RBDSDE, with \( t \in [0, T] \)

\[
Y_t = \xi + \int_t^T E'f(s, \omega, \omega', Y_s, y_s', Z_s, z_s') ds + \int_t^T E'g(s, \omega, \omega', Y_s, y_s', Z_s, z_s') dB_s
+ K_T - K_t - \int_t^T Z_s dW_s.
\]

According to Theorem 1 in Bahlali et al. [2], there exists a unique solution \((Y, Z) \in S^2 \times M^2\)

i.e., if we define the process

\[
K_t = Y_0 - Y_t - \int_0^t E'f(s, \omega, \omega', Y_s, y_s', Z_s, z_s') ds
- \int_0^t E'g(s, \omega, \omega', Y_s, y_s', Z_s, z_s') dB_s + \int_0^t Z_s dW_s,
\]

then \((Y, Z, K)\) satisfies Definition 2.1. Hence, if we define \( \Theta(y, z) = (Y, Z) \), then \( \Theta \) maps \( S^2 \times M^2 \) itself. We show now that \( \Theta \) is contractive. To this end, take any \((y^i, z^i) \in S^2 \times M^2\) \((i = 1, 2)\), and let \( \Theta(y^i, z^i) = (Y^i, Z^i) \).

We denote \((\bar{Y}, \bar{Z}, \bar{K}) = (Y^1 - Y^2, Z^1 - Z^2, K^1 - K^2)\) and \((\bar{y}, \bar{z}) = (y^1 - y^2, z^1 - z^2)\).

Therefore, Itô’s formula applied to \( |\bar{Y}|^2 e^{\beta t} \) where \( \beta > 0 \), and the inequality \( 2ab \leq \left( \frac{1}{\alpha} \right) a^2 + \delta b^2 \), lead to

\[
E|\bar{Y}_t|^2 e^{\beta t} + \left( \beta - 3L - \frac{8L^2}{1 - 2\alpha} \right) E \int_t^T |\bar{Z}_s|^2 e^{\beta s} ds + \frac{1}{2} E \int_t^T e^{\beta s} |\bar{Z}_s|^2 ds
\]

\[
\leq E \int_t^T e^{\beta s} |\bar{Y}_s| (dK^1_s - dK^2_s) 
+ E \int_t^T e^{\beta s} \left( \left( L + \frac{1 - 2\alpha}{2L} \right) |\bar{Y}_s|^2 + \left( \frac{1 + 2\alpha}{4} \right) |\bar{Z}_s|^2 \right) ds
\]

Choosing \( \beta = 3L + \frac{8L^2}{1 - 2\alpha} + \frac{1}{2} \left( \frac{4}{1 + 2\alpha} \right) \left( L + \frac{1 - 2\alpha}{2L} \right) \) and setting \( M = \left( \frac{4}{1 + 2\alpha} \right) \left( L + \frac{1 - 2\alpha}{2L} \right) \)

yield
(a) For almost every Lemma 4 can be proved by adapting the proof given in Alibert and Bahlali [1].

Towards this end, we consider the following assumption.

\[ \|Y\|_\rho = \left( E \int_t^T e^{\beta s} \left( |\overline{Y}_s| + |Z_s| \right) ds \right)^{\frac{1}{2}}. \]

Moreover, it has a unique fixed point, which is the unique solution of the MF-RBDSDE with data \((\xi, f, g, S)\).

4. RBDSDEs With a Continuous Coefficient

In this section we prove the existence of a solution to the MF-RBDSDE where the coefficient is only continuous.

Towards this end, we consider the following assumption.

**H5**

i) For a.e \((t, w)\), the mapping \((y, y', z, z') \mapsto f(t, y, y', z, z')\) is continuous. ii) There exist constants \(L > 0\) and \(\alpha \in \left(0, \frac{1}{2}\right)\), such that for every \((t, \omega) \in \Omega \times [0, T]\) and \((y, z, y', z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d\),

\[
\begin{align*}
|f(t, y, y', z, z')| &\leq L(1 + |y| + |y'| + |z| + |z'|) \\
|g(t, y_1, y'_1, z_1, z'_1) - g(t, y_2, y'_2, z_2, z'_2)| &\leq L\left( |y'_1 - y'_2|^2 + |y_1 - y_2|^2 \right) + \alpha \left( |z'_1 - z'_2|^2 + |z_1 - z_2|^2 \right)
\end{align*}
\]

**Theorem 4.1.** Under assumption H1), H3), H4) and H5), the MF-RBDSDE (1) has an adapted solution \((Y, Z, K)\) which is a minimal one, in the sense that, if \((Y^*, Z^*)\) is any other solution we have \(Y \leq Y^*, P - a.s.\)

Before giving a proof to this theorem, we invoke first the following classical lemma, which can be proved by adapting the proof given in Alibert and Bahlali [1].

**Lemma 4.1.** Let \(f : [0, T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}\) be a measurable function such that:

(a) For almost every \((t, \omega) \in [0, T] \times \overline{\Omega}, x \mapsto f(t, \omega, x)\) is continuous,

(b) There exists a constant \(K > 0\) such that for every \((t, y', y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d\).
\[ |f(t,y',z)| \leq K(1 + |y'| + |y| + |z|) \text{ a.s.} \]

(c) For almost every \(y, z, f(t,y',y,z)\) is increasing in \(y'\).

Then, the sequence of functions \(f_n(t,y',y,z) = \inf_{(u,v,w) \in \mathbb{Q}^{2d}} \{f(t,u,v,w) + n(y' - u) + n|y - v| + n|z - w|\}\) is well defined for each \(n \geq K\) and satisfies:

(1) for every \((t,y',y,z) \in [0,T] \times R^{2d}\), \(f_n(t,y',y,z) \leq K(1 + |y'| + |y| + |z|)\).

(2) for every \((t,y',y,z) \in [0,T] \times R^{2d}\), \(n \to f_n(t,x)\) is increasing,

(3) for every \((t,y',y,z) \in [0,T] \times R^{2d}\), \(y' \to f_n(t,y',y,z)\) is increasing,

(4) for every \(n \geq K\), \((t,y',y,z) \in [0,T] \times R^{2d}\), \(f_n(t,y'^1,y^1,z^1) - f_n(t,y'^2,y^2,z^2) \leq n(|y'^1 - y'^2| + |y^1 - y^2| + |z'^1 - z^2|)\).

(5) If \((y'_n, y_n, z_n) \to (y', y, z)\), as \(n \to \infty\) then for every \(t \in [0,T] f_n(t,y'_n, y_n, z_n) \to f(t,y',y,z)\) as \(n \to \infty\).

Since \(\xi\) satisfies H3, we get from theorem 3.1, that for every \(n \in N^*\), there exists a unique solution \(\{Y^n_t, Z^n_t, K^n_t\}, 0 \leq t \leq T\) for the following MF-RBDSDE

\[
\begin{align*}
Y^n_t &= \xi + \int_t^T f_n(s, Y^n_s, Z^n_s)ds + K^n_t - K^n_t + \int_t^T g(s, Y^n_s, Z^n_s)dB_s \\
&\quad - \int_t^T Z^n_s dB_s, \quad 0 \leq t \leq T, \\
Y^n_t \geq S_t, \quad \int_0^T (Y^n_t - S_t)dK^n_s = 0.
\end{align*}
\]

Since, \(|f^1(t,u,v,w) - f^1(t,u',v',w')| \leq L(|u - u'| + |v - v'| + |w - w'|)\), we consider the function defined by \(f^1(t,u,v,w) := L(1 + |u| + |v| + |w|)\),

then a similar argument as before shows that there exists a unique solution \(((U_s, V_s, K_s), 0 \leq s \leq T)\) to the following MF-RBDSDE:

\[
\begin{align*}
U_t &= \xi + \int_t^T f^1(s, U_s, U_s, V_s)ds + K_T - K_t + \int_t^T g(s, U_s, U_s, V_s)dB_s - \int_t^T V_s dB_s \\
&\quad - \int_t^T Z^n_s dB_s, \quad 0 \leq t \leq T, \\
&\quad \int_0^T (U_s - S_t)dK_s = 0.
\end{align*}
\]

We would also need the following comparison theorem.

**Theorem 4.2.** (Comparison theorem) Let \((\xi^1, f^1, g, S^1)\) and \((\xi^2, f^2, g, S^2)\) be two MF-RBDSDEs. Each one satisfying all the previous assumptions H1), H2), H3) and H4). Assume moreover that:

i) \(\xi^1 \leq \xi^2\) a.s.

ii) \(f^1(t,y',y,z',z) \leq f^2(t,y',y,z',z)\) dP \(\times dt\) a.e. \(\forall (y',y,z',z) \in R \times R^d\).

iii) \(S^1_t \leq S^2_t, 0 \leq t \leq T\) a.s.

Let \((Y^1, Z^1, K^1)\) be a solution of MF-RBDSDE \((\xi^1, f^1, g, S^1)\) and \((Y^2, Z^2, K^2)\) be a solution of MF-RBDSDE \((\xi^2, f^2, g, S^2)\). We suppose also:

a) One of the two generators is independent of \(z'\).
b) One of the two generators is nondecreasing in \( y' \).

then

\[ Y_t^1 \leq Y_t^2, \quad 0 \leq t \leq T \quad \text{a.s.} \]

**Proof.** Suppose that (a) is satisfied by \( f^1 \) and (b) by \( f^2 \). Applying Itô’s formula to \( |(Y_t^1 - Y_t^2)^+|^2 \), and passing to expectation, we have

\[
E[(Y_t^1 - Y_t^2)^+|^2 + E \int_t^T 1_{\{Y_s^1 > Y_s^2\}} |Z_s^1 - Z_s^2|^2 ds
\]

\[
= 2 E \int_t^T (Y_s^1 - Y_s^2)^+ E' \left( f^1 \left( s, (Y_s^1)', Y_s^1, Z_s^1 \right) - f^2 \left( s, (Y_s^2)', Y_s^2, (Z_s^2)', Z_s^2 \right) \right) ds
\]

\[
+ 2 E \int_t^T (Y_s^1 - Y_s^2)^+ dK_s^1 - dK_s^2
\]

\[
+ E \int_t^T E' \left( g \left( s, (Y_s^1)', Y_s^1, Z_s^1 \right) - g \left( s, (Y_s^2)', Y_s^2, (Z_s^2)', Z_s^2 \right) \right) |Z_s^1 - Z_s^2|^2 ds
\]

Since on the set \( \{ Y_s^1 > Y_s^2 \} \), we have \( Y_t^1 > S_t^2 \geq S_t^1 \), then

\[
\int_t^T (Y_s^1 - Y_s^2)^+ (dK_s^1 - dK_s^2) = - \int_t^T (Y_s^1 - Y_s^2)^+ dK_s^2 \leq 0.
\]

Since \( f^1 \) and \( f^2 \) are Lipschitzian, we have on the set \( \{ Y_s > Y_s' \} \),

\[
E[(Y_t^1 - Y_t^2)^+|^2 + E \int_t^T 1_{\{Y_s^1 > Y_s^2\}} |Z_s^1 - Z_s^2|^2 ds
\]

\[
\leq E \int_t^T \left( (6L + \frac{L^2}{1-\alpha}) |(Y_t^1 - Y_t^2)^+|^2 + |Z_s^1 - Z_s^2|^2 \right) ds,
\]

then

\[
E[(Y_t^1 - Y_t^2)^+|^2 \leq E \int_t^T \left( 6L + \frac{L^2}{1-\alpha} \right) |(Y_t^1 - Y_t^2)^+|^2 ds.
\]

The required result follows by using Gronwall’s lemma.

**Lemma 4.2.** i) a.s. for all \( t \), \( Y_t^0 \leq Y_t^n \leq Y_t^{n+1} \leq U_t \). ii) There exists \( Z \in M^2 \), such that \( Z^n \) converges to \( Z \).

**Proof.** Assertion i) follows from the comparison theorem. We therefore need to prove ii) only.

Itô’s formula yields

\[
E|Y_0^n|^2 + E \int_0^T |Z^n_s|^2 ds = E|\xi|^2 + 2E \int_0^T Y_t^n E' \left( f_n \left( s, (Y_t^n)', Y_t^n, Z_t^n \right) \right) ds + 2E \int_0^T S_s dK^n_s
\]

\[
+ E \int_0^T E' \left( |g(s, (Y_t^n)', Y_t^n, Z_t^n)|^2 \right) ds.
\]

From assumption H5), and the inequality \( 2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2 \) for \( \varepsilon > 0 \), we get:
\[ E \int_0^T |Z^n_s| ^2 ds \leq E|\xi|^2 + \frac{LT}{\varepsilon} + E \int_0^T |g(s,0,0,0)|^2 ds + (3L\varepsilon + \frac{L}{\varepsilon} + 4L)E \int_0^T |Y^n_s| ^2 ds \]
\[ + \left( \frac{L}{\varepsilon} + \alpha \right) E \int_0^T \frac{1}{\varepsilon} |Z^n_s|^2 ds + 2E \int_0^T S_s dK^n_s. \]

On the other hand, we have from (2)
\[ K^n_s = Y^n_0 - \zeta - \int_0^T E'f_n(s, (Y^n_s)', Y^n_s, Z^n_s) ds - \int_0^T E'g(s, (Y^n_s)', Y^n_s, Z^n_s) dB_s \]
\[ + \int_0^T Z^n_s dW_s. \quad (4) \]

Then
\[ E(K^n_s)^2 \leq C \left( 1 + E \int_0^T \| Z^n_s \|^2 ds \right). \]

We also have
\[ 2E \int_0^T S_s dK^n_s \leq \frac{1}{\beta} E \left( \sup_t |S_t|^2 \right) + \beta E(K^n_s)^2 \]
\[ \leq \frac{1}{\beta} E \left( \sup_t |S_t|^2 \right) + \beta C \left( 1 + E \int_0^T \| Z^n_s \|^2 ds \right), \]

which leads to
\[ E \int_0^T |Z^n_s|^2 ds \leq E|\xi|^2 + \frac{LT}{\varepsilon} + \beta C + E \int_0^T |g(s,0,0,0)|^2 ds + (3L\varepsilon + \frac{L}{\varepsilon} + 4L)E \int_0^T |Y^n_s|^2 ds \]
\[ + \left( \frac{L}{\varepsilon} + \alpha + \beta C \right) E \int_0^T \frac{1}{\varepsilon} |Z^n_s|^2 ds + \frac{1}{\beta} E \left( \sup_t |S_t|^2 \right). \]

Choosing \( \varepsilon, \beta \) such that \( \left( \frac{L}{\varepsilon} + \alpha + \beta C \right) < 1 \), we obtain
\[ E \int_0^T \| Z^n_s \|^2 ds \leq C. \]

For \( n, p \geq K \), Itô’s formula gives,
\[ E(Y^n_0 - Y^p_0)^2 + E \int_0^T \| Z^n_s - Z^p_s \|^2 ds \]
\[ = 2E \int_0^T (Y^n_s - Y^p_s)E'(f_n(s, Y^n_s', Y^n_s, Z^n_s) - f_p(s, Y^p_s, (Y^n_s)', Z^n_s)) ds \]
\[ + 2E \int_0^T (Y^n_s - Y^p_s) dK^n_s + 2E \int_0^T (Y^n_p - Y^p_p) dK^p_s \]
\[ + E \int_0^T \| E'(g(s, Y^n_s', (Y^n_s)', Z^n_s) - g(s, Y^p_s, (Y^n_s)', Z^n_s)) \|^2 ds. \]

But
\[ E \int_0^T (Y^n_s - Y^p_s) dK^n_s = E \int_0^T (S_s - Y^p_s) dK^p_s \leq 0. \]

Similarly, we have \( E \int_0^T (Y^n_p - Y^p_p) dK^p_s \leq 0. \) Therefore,
Then, by using Lemma 4.1, we get
\[
E \int_0^T \|Z_s^n - Z_s^p\|^2 ds \leq 2E \int_0^T (Y_s^n - Y_s^p)E'f_n(s, Y_s^n, (Y_s^n)', Z_s^n) - f_p(s, Y_s^p, (Y_s^p)', Z_s^p)) ds \\
+ E \int_0^T \|E'(g(s, Y_s^n, (Y_s^n)', Z_s^n) - g(s, Y_s^p, (Y_s^p)', Z_s^p))\|^2 ds.
\]

By Hölder’s inequality and the fact that \(g\) is Lipschitzian, we get
\[
E \int_0^T \|Z_s^n - Z_s^p\|^2 ds \\
\leq \left( E \int_0^T (Y_s^n - Y_s^p)^2 ds \right)^{1/2} \left( E \int_0^T E'(f_n(s, Y_s^n, (Y_s^n)', Z_s^n) - f_p(s, Y_s^p, (Y_s^p)', Z_s^p))^2 ds \right)^{1/2} \\
+ LE \int_0^T \left( |Y_s^n - Y_s^p|^2 + |(Y_s^n)\prime - (Y_s^p)\prime|^2 \right) ds + \alpha E \int_0^T |Z_s^n - Z_s^p|^2 ds
\]

Since \( \sup_n E \int_0^T |f_n(s, Y_s^n, (Y_s^n)', Z_s^n)|^2 \leq C \), we obtain,
\[
E \int_0^T \|Z_s^n - Z_s^p\|^2 ds \leq C \left( E \int_0^T (Y_s^n - Y_s^p)^2 ds \right)^{1/2}.
\]

Hence
\[
E \int_0^T \|Z_s^n - Z_s^p\|^2 ds \to 0, \text{ as } n, p \to \infty.
\]

Thus \((Z^n)_{n \geq 1}\) is a Cauchy sequence in \(M^2(\mathbb{R}^d)\).

\[\blacksquare\]

### 4.1. Proof of Theorem 4.1.

Let \(Y_t = \sup_n Y_t^n\), and we have \((Y^n, Z^n) \to (Y, Z)\) in \(S^2(\mathbb{R}^d) \times M^2(\mathbb{R}^d)\). Then, along a subsequence which we will still denote as \((Y^n, Z^n)\), we have
\((Y^n, Z^n) \to (Y, Z), \quad dt \otimes dP \text{ a.e.}\)

Then, by using Lemma 4.1, we get \(f_n(t, Y_t^n, (Y_t^n)\prime, Z_t^n) \to f(t, Y_t, (Y_t)\prime, Z_t) \quad dP dt \text{ a.e.}\). On the other hand, since \(Z^n \to Z\) in \(M^2(\mathbb{R}^d)\), then there exists an \(\Lambda \in M^2(\mathbb{R})\) and a subsequence, which we continue to denotes as denote \(Z^n\), such that \(\forall n, |Z^n| \leq \Lambda, Z^n \to Z, dt \otimes dP \text{ a.e.}\)

Moreover, from H5), and Lemma 4.2, we have
\[
|f_n(t, Y_t^n, (Y_t^n)\prime, Z_t^n)| \leq \kappa(1 + \sup_n |Y_t^n| + \sup_n |(Y_t^n)\prime| + \Lambda(t)) \in L^2([0, T], dt), \quad P - a.s.
\]

It follows from the dominated convergence theorem that,
\[
E \int_0^T |E'(f_n(s, Y_s^n, (Y_s^n)\prime, Z_s^n) - f(s, Y_s, (Y_s)\prime, Z_s))|^2 ds \to 0, \quad n \to \infty.
\]

Subsequently,
\[ E \int_0^T \| E'(g(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)) \|^2 ds \]
\[ \leq CE \int_0^T E' \left( |Y^n_s - Y_s|^2 + |(Y^n'_s) - (Y'_s)|^2 \right) ds \]
\[ + \alpha E \int_0^T \| Z^n_s - Z_s \|^2 ds \rightarrow 0, \quad \text{as } n \rightarrow \infty. \]

It is not difficult to show that \((Y, Z)\) is a solution to our MF-RBDSDE. Indeed, let

\[ Y_t = \xi + \int_t^T E'f(s, Y_s, (Y'_s), Z_s) ds + K_T - K_t \]
\[ + \int_t^T E'g(s, Y_s, (Y'_s), Z_s) dB_s - \int_t^T Z_s dW_s, \]

where \(Z \in M^2, Y \in S^2, K_T \in L^2, Y_t \geq S_t, (K_t)\) is continuous and nondecreasing, \(K_0 = 0\) and \(\int_0^T (Y_t - S_t) dK_t = 0\), and \((Y^*, Z^*, K^*)\) be a solution of (1). Then, by theorem 4.2, we have for every \(n \in \mathbb{N}^*, Y^n \leq Y^*\). Therefore, \(Y\) is a minimal solution of (1).

Remark 4.1. Using the same arguments and the following approximating sequence

\[ \mathcal{E}h_n(t,x,y,z) = \sup_{(u,v,w)\in Q^p} \{ h(u,v,w) - n|x - u| - n|v - y| - n|z - w| \}, \]

one can prove that the MF-RBDSDE (1) has a maximal solution.

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References


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