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A Unified Approach to Generate Weighted Newton Third Order Methods for Solving Nonlinear Equations

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Abstract. *In this paper, we present an easily applicable scheme for constructing one-point third order iterative formulae for the computation of real or complex solutions of nonlinear equations. The scheme is powerful and interesting since it regenerates almost all available one-point third order methods in literature. For example, Laguerre, Chebyshev, Euler, Halley, Ostrowski, Hansen-Patrick, and super-Halley methods, etc. Moreover, many new methods can in principle be generated. A convergence analysis is also provided to establish a third-order convergence of the proposed scheme. In order to support the theory developed in this paper, some numerical tests are performed.*

Key words : Nonlinear Equations, Iterative Methods, Newton's Method, Root Finding, Order of Convergence.

AMS Subject Classifications : 65H05

1. Introduction

Solving non-linear equations is a common and important problem in science and engineering [1, 2]. Analytic methods for solving such equations are almost non-existent and therefore it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedures. With the advancement of computers, the problem of solving non-linear equations by numerical methods has gained more importance than ever before.

In this paper, we consider iterative methods for finding a simple root r of the nonlinear equation $f(x) = 0$, where $f(x)$ is a real or complex analytic function. Newton's method is undoubtedly the most widely used algorithm for solving such equations, which starts with an initial approximation x_0 closer to the root r and generates a sequence of successive iterates

$\{x_n\}_0^\infty$ converging quadratically to the root. It is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots \tag{1}$$

In order to improve the order of convergence of Newton's method, a number of third order methods have been proposed at the expense of an additional evaluation of the second derivative. For example, Halley [3, 4], Euler [4], Ostrowski's square root method [5], Chebyshev [6], Hansen-Patrick [7], Laguerre [5], super-Halley [8], Chun [9], Jiang-Han [10] etc. are well-known methods requiring the evaluation of f , f' and f'' per step. In the terminology of Traub [6] such methods are classified as one-point iterative methods without memory. It is observed that most of the above methods are based on considering appropriate quadratic curve approximation (see[8-10]).

We shall be concerned with developing a unified scheme to obtain one-point methods without memory with at least cubic convergence. Unlike the approaches which have been used in deriving the existing methods, our approach is based on a simple modification of Newton's method (1). The proposed scheme is interesting since it regenerates all the above mentioned well-known methods. In addition, many new methods can in principle be obtained from it.

The paper is organized as follows. Some basic definitions relevant to the present work are presented in Section 2. In Section 3, the scheme is proposed and its behavior is analyzed. Section 4 contains a development of the forms for various new methods. In Section 5, the new methods are tested and compared with other well-known methods on a number of problems. Concluding remarks are given in Section 6.

2. Basic Definitions

Definition 2.1. Let $f(x)$ be a real or complex function with a simple root r and let $\{x_n\}_{n\in N}$ be a sequence of real or complex numbers that converges towards r. Then, we say that the order of convergence of the sequence is p if there exists a $p \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \frac{x_{n+1}-r}{(x_n-r)^p} = C, \tag{2}$$

for some $C \neq 0$, where C is known as the asymptotic error constant. If $p = 1, 2$ or 3, the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Definition 2.2. Let $e_n = x_n - r$ be the error in the n^{th} iteration, we call the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}), \tag{3}$$

the error equation. If we can obtain an error equation for any iterative method, then the value of p is the order of its convergence.

Definition 2.3. Let θ be the number of new pieces of information required by a method. A 'piece of information' is typically any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [11] and is defined by

$$E = p^{1/\theta}, \tag{4}$$

where p is the order of convergence of the method.

Definition 2.4. Suppose that x_{n+1}, x_n and x_{n-1} are three successively close iterations to the root r. The computational order of convergence ρ (see[12]) is approximated by

$$\rho \cong \frac{\ln |(x_{n+1}-r)/(x_n-r)|}{\ln|(x_n-r)/(x_{n-1}-r)|}, \tag{5}$$

which pertains to the error equation (3).

3. The Scheme and its Properties

Our aim here is to develop an iterative scheme that improves the order of convergence of Newton's method by following the scheme

$$x_{n+1} = x_n - W(L_n)\frac{f(x_n)}{f'(x_n)}, \tag{6}$$

where $L_n = L(x_n) = \dfrac{f(x_n)f''(x_n)}{[f'(x_n)]^2}$, and $W(t)$ is called a weight function. For a reason, to be clarified later, this scheme is called weighted Newton iterative scheme. The nature and properties of (6) are illustrated by the theorem that follows.

Theorem 3.1. *Let $f(x)$ be a real or complex function. Assuming that $f(x)$ is sufficiently smooth in some neighborhood of a zero, r, say. Further, assume that $f'(r) \neq 0$ and x_0 is sufficiently close to r. Then, the iteration process converges cubically to the root r, provided that $W(0) = 1, W'(0) = \frac{1}{2}$ and $|W''(0)| < \infty$.*

Proof. Using Taylor expansion of $f(x_n)$ and $f'(x_n)$ about r and using the fact that $f(r) = 0$, $f'(r) \neq 0$, we have

$$f(x_n) = f'(r)[e_n + C_2e_n^2 + C_3e_n^3 + C_4e_n^4 + O(e_n^5)], \tag{7}$$

where $e_n = x_n - r$ and $C_k = \dfrac{1}{k!}\dfrac{f^{(k)}(r)}{f'(r)}, \quad k = 2, 3, \ldots$

$$f'(x_n) = f'(r)[1 + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3 + O(e_n^4)]. \tag{8}$$

and

$$f''(x_n) = f'(r)[2C_2 + 6C_3e_n + 12C_4e_n^2 + O(e_n^3)]. \tag{9}$$

Also

$$[f'(x_n)]^2 = [f'(r)]^2\ [1 + 4C_2e_n + 2(2C_2^2 + 3C_3)e_n^2 + 4(3C_2C_3 + 2C_4)e_n^3 + O(e_n^4)], \tag{10}$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - C_2e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4). \tag{11}$$

Using (7), (9) and (10), we get

$$L_n = \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2} = 2C_2e_n - 6(C_2^2 - C_3)e_n^2 + 4(4C_2^3 - 7C_2C_3 + 3C_4)e_n^3 + O(e_n^4), \quad (12)$$

Using theTaylor′s expansion of $W(L_n)$ leads to

$$W(L_n) = W(L(r)) + (L_n - L(r))W'(L(r)) + \tfrac{1}{2}(L_n - L(r))^2W''(L(r)) + O((L_n - L(r))^3). \quad (13)$$

Since $L(r) = 0,$ therefore

$$W(L_n) = W(0) + L_nW'(0) + \tfrac{1}{2}L_n^2W''(0) + O((L_n)^3). \quad (14)$$

Using (12), we have

$$W(L_n) = W(0) + 2C_2W'(0)e_n - 2(3(C_2^2 - C_3)W'(0) - C_2^2W''(0))e_n^2 + O(e_n^3). \quad (15)$$

Then invoke (6), (11) and (15) to obtain the error equation as

$$e_{n+1} = (1 - W(0))e_n + C_2(W(0) - 2W'(0))e_n^2 - 2((C_2^2 - C_3)W(0) + (-4C_2^2 + 3C_3)W'(0) + C_2^2W''(0))e_n^3 + O(e_n^4).$$

We see then that if W is any function with $W(0) = 1, W'(0) = \frac{1}{2}$ and $|W''(0)| < \infty,$ then the convergence order of the iterative process (6) is three, and hence the error equation is

$$e_{n+1} = (2(1 - W''(0))C_2^2 - C_3)e_n^3 + O(e_n^4). \quad (16)$$

This completes the proof. ■

4. Some Concrete Weights

In this section we consider various forms of the weight function $W(t)$ satisfying the conditions of theorem 3.1. Based on these forms several third order methods are established. The existing methods are then shown as special forms of the obtained methods. In the sequel, α and β are some fixed parameters.

Method 1. For the function $W,$ defined by

$$W(t) = 1 + \tfrac{1}{2}t + \alpha t^2,$$

it is clear that the conditions of theorem 3.1 are satisfied. Hence we get the one-parameter family of third order methods:

$$x_{n+1} = x_n - (1 + \tfrac{1}{2}L_n + \alpha L_n^2)\frac{f(x_n)}{f'(x_n)}. \quad (17)$$

The error equation (16) now takes the form

$$e_{n+1} = (2(1 - 2\alpha)C_2^2 - C_3)e_n^3 + O(e_n^4).$$

If we consider $\alpha = 0,$ we obtain Chebyshev's method [6]

$$x_{n+1} = x_n - (1 + \tfrac{1}{2}L_n)\frac{f(x_n)}{f'(x_n)}. \quad (18)$$

Method 2. The use of a weight function W, defined by

$$W(t) = \frac{2}{2-t+\alpha t^2},$$

leads also to satisfaction of the conditions of theorem 3.1. A case that duplicates Jiang-Han rational iterative family [10]

$$x_{n+1} = x_n - \frac{2}{2-L_n+\alpha L_n^2}\frac{f(x_n)}{f'(x_n)}. \tag{19}$$

The error equation (16) now takes the form

$$e_{n+1} = (2(1+2\alpha)C_2^2 - C_3)e_n^3 + O(e_n^4).$$

Method 3. The function W, defined by

$$W(t) = 1 + \frac{1}{2}\frac{t}{1-\alpha t},$$

also satisfies the conditions of theorem 3.1. It leads to a one-parameter family of third order methods

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_n}{1-\alpha L_n}\right)\frac{f(x_n)}{f'(x_n)}. \tag{20}$$

The associated error equation (16) is

$$e_{n+1} = (2(1-\alpha)C_2^2 - C_3)e_n^3 + O(e_n^4).$$

The family includes the following methods as the special cases. For $\alpha = 1$, we obtain a super-Halley method [8],

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_n}{1-L_n}\right)\frac{f(x_n)}{f'(x_n)}. \tag{21}$$

If we let

$$\alpha = \frac{1}{2(1+\beta f'^2(x_n))}, \beta \in R,$$

we obtain Chun's family [9]

$$x_{n+1} = x_n - \frac{2(1+\beta f'^2(x_n))+\beta L_n f'^2(x_n)}{2(1+\beta f'^2(x_n))-L_n}\frac{f(x_n)}{f'(x_n)}. \tag{22}$$

Method 4. For

$$W(t) = \frac{\alpha+1}{\alpha+\left(1-\frac{\alpha+1}{\beta}t\right)^{\beta/2}}, \beta \neq 0,$$

it can be shown that the conditions of theorem 3.1 are satisfied. Thus, we get a two-parameter family of third order methods

$$x_{n+1} = x_n - \frac{\alpha+1}{\alpha+\left(1-\frac{\alpha+1}{\beta}t\right)^{\beta/2}} \frac{f(x_n)}{f'(x_n)}, \tag{23}$$

which satisfies the error equation

$$e_{n+1} = \left[\left(1+\frac{(\beta-2)(\alpha+1)}{2\beta}\right)C_2^2 - C_3\right]e_n^3 + O(e_n^4).$$

The family (23) contains the following as special cases. When $\alpha = \beta = 1$, one obtains Euler's method [3]

$$x_{n+1} = x_n - \frac{2}{1+(1-2L_n)^{1/2}} \frac{f(x_n)}{f'(x_n)}, \tag{24}$$

whereas the case $\alpha = 1, \beta = 2$ duplicates Halley's method [3, 4]

$$x_{n+1} = x_n - \frac{2}{2-L_n} \frac{f(x_n)}{f'(x_n)}. \tag{25}$$

When $\alpha = 0,\ \beta = 1,$ one obtains Ostrowski's square root method [5]

$$x_{n+1} = x_n - \frac{1}{(1-L_n)^{1/2}} \frac{f(x_n)}{f'(x_n)}. \tag{26}$$

The case of $\beta = 1$ is Hansen-Patrick family [7]

$$x_{n+1} = x_n - \frac{\alpha+1}{\alpha+[1-(\alpha+1)L_n]^{1/2}} \frac{f(x_n)}{f'(x_n)}. \tag{27}$$

Finally, if $f(x)$ is a polynomial of degree n and if we let $\alpha = 1/(n-1), \beta = 1,$ where $n \neq 1$, we obtain the Laguerre method [5]

$$x_{n+1} = x_n - \frac{n}{1+[(n-1)^2-n(n-1)L_n]^{1/2}} \frac{f(x_n)}{f'(x_n)}. \tag{28}$$

Method 5. For the function W, defined by

$$W(t) = \frac{2}{1-\alpha t \pm \left(1+2\frac{(\alpha-1)}{\beta}t+\frac{\alpha^2}{\beta}t^2\right)^{\beta/2}}, \beta \neq 0,$$

it is clear that the conditions of theorem 3.1 are satisfied. Hence, we have another two-parameter family of third order methods

$$x_{n+1} = x_n - \frac{2}{1-\alpha L_n \pm \left(1+2\frac{(\alpha-1)}{\beta}L_n+\frac{\alpha^2}{\beta}L_n^2\right)^{\beta/2}} \frac{f(x_n)}{f'(x_n)}. \tag{29}$$

The error equation (16) for this family is given by

$$e_{n+1} = \left[\left(1-\frac{\alpha 2}{2}+\frac{(\beta-2)(\alpha-1)2}{2\beta}\right)C_2^2 - C_3\right]e_n^3 + (e_n^4).$$

This family also includes remarkable special cases. If we let $\alpha = 0$ and $\beta = 2,$ we obtain Halley's method [3,4]; while for $\beta = 1,$ we obtain Jiang-Han irrational iterative family [10]

$$x_{n+1} = x_n - \frac{2}{1-\alpha L_n \pm [1+2(\alpha-1)L_n+\alpha^2 L_n^2]^{1/2}} \frac{f(x_n)}{f'(x_n)}. \tag{30}$$

5. Numerical Results

Now we present some numerical tests for various cubically convergent iterative schemes and Newton method. Compared are the Newton method (NM) defined by equation (1), Chebyshev method (CM) defined by (18), Jiang-Han rational method (JHM) defined by (19) with parameter $\alpha = 1,$ super-Halley method (SH) defined by (21), Euler method (EM) defined by (24), Halley method (HM) defined by (25) and Ostrowski square root method (OM) defined by (26). To represent new methods of the scheme (6), we choose three methods designated as M1, M2 and M3, where M1 is defined by the equation (17) with $\alpha = 0.5,$ M2 is defined by (23) with $\alpha = 0.5, \beta = 1,$ and M3 by (29) with $\alpha = \beta = 0.5.$

We conceive an approximate solution rather than the exact root, depending on the computer precision $(\epsilon).$ The stopping criteria used for computer program are: (a) $|x_{n+1} - x_n| < \epsilon,$ and (b) $|f(x_{n+1})| < \epsilon.$ Hence when the stopping criterion is satisfied, x_{n+1} is taken as the computed root r. It is well-known that convergence of an iterative formula is guaranteed only when the initial approximation is sufficiently close to the root. However, for a given nonlinear equation, it is rather hard to choose an initial approximation near a root. In general moreover, any iterative scheme may be divergent when the initial approximation is far from the root.

The test functions and root $r,$ correct up to 16 decimal places are displayed in Table 1. The first seven functions we have selected are the same as in [12], whereas the last is selected from [7]. Table 2 shows the values of the initial approximation (x_0) chosen from both ends of the root and the number of iterations (n) that is required to approximate the root, and the computational order of convergence (ρ) defined by (5). For the numerical results of Table 2 we use a fixed stopping criterion of $\epsilon = 0.5 \times 10^{-16}$.

In Table 3, we exhibit the absolute values of the error e_n calculated for the same total number of function evaluations (TFE) for each method. The TFE is counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivatives. In the calculations, 12 TFE are used by each method. That means 6 iterations are used for NM and 4 iterations for the remaining methods. It is quite obvious that increasing the order of the method leads to more precision. For this reason and for more consistency in comparisons, all computations of Table 3 are done with multiprecision arithmetic using 300 significant digits.

It can be observed that the computed results, displayed in Table 2, overwhelmingly support the theory of convergence and the efficiency analysis discussed in Section 3. Table 3 also shows that in most of the cases, the results obtained with our new methods are similar to the other existing methods. Moreover, it can be seen that in some problems the accuracy of presented methods is higher than the respective competitors in terms of the number of significant digits gained by each method. The methods, however behave similarly, i.e. for some initial guesses one is better while for other initial approximations the another one would be more appropriate. In all, we can infer that the convergence behavior of the considered one-point methods strongly depends on the structure of tested functions and the accuracy of starting points. We should also remark that one-point iterative methods without memory of the same order and the same computational cost show a similar convergence behavior and produce results of roughly the same accuracy.

Table 1: Test functions

$f(x)$			Root (r)
$f_1(x)$	=	$x^3 + 4x^2 - 10$	1 : 3652300134140968
$f_2(x)$	=	$sin^2(x) - x^2 + 1$	1 : 4044916482153412
$f_3(x)$	=	$x^2 - e^x - 3x + 2$	0 : 2575302854398608
$f_4(x)$	=	$cos(x) - x$	0 : 7390851332151606
$f_5(x)$	=	$(x - 1)^3 - 1$	2 : 0000000000000000
$f_6(x)$	=	$x^3 - 10$	2 : 1544346900318837
$f_7(x)$	=	$xe^{x^2} - sin^2(x) + 3cos(x) + 5$	–1 : 207647827130919
$f_8(x)$	=	$x^6 - 4x^5 - 5x^4 + 190x^3 - 666x^2 + 944x - 600$	$1 \pm i$

Table 2: Performance of the methods

$f(x)$	x_0	n										ρ									
		NM	CM	JHM	SH	EM	HM	OM	M1	M2	M3	NM	CM	JHM	SH	EM	HM	OM	M1	M2	M3
f_1	1	5	4	4	3	3	3	3	3	2	3	2	3	3	3	3	3	3	3	3	3
	2	5	4	4	3	3	4	3	3	3	3	2	3	3	3	3	3	3	3	3	3
f_2	1.2	5	4	4	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
	2	6	4	4	3	3	4	4	3	3	4	2	3	3	3	3	3	3	3	3	3
f_3	0	4	3	3	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
	1	4	3	3	4	4	3	4	4	4	3	2	3	3	3	3	3	3	3	3	3
f_4	0.5	4	3	3	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
	1	4	3	3	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
f_5	1.8	5	4	4	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
	2.5	6	4	4	3	4	4	3	3	3	3	2	3	3	3	3	3	3	3	3	3
f_6	2	4	3	3	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
	2.5	5	3	3	3	3	4	3	3	3	3	2	3	3	3	3	3	3	3	3	3
f_7	-1	6	4	4	4	4	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
	-1.3	5	3	3	3	3	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3
f_8	$.5 \pm 1.5i$	6	4	4	4	3	4	4	4	4	4	2	3	3	3	3	3	3	3	3	3

Table 3: Error for the same total number of function evaluations (TFE = 12) for all methods.

$f(x)$	x_0	$\|e_n\| = \|x_n - r\|$				
		NM	CM	JHM	SH	EM
f_1	1	2.41e - 44	1.81e - 42	4.94e - 45	1.50e - 76	1.34e - 84
	2	7.49e - 39	3.75e - 42	2.70e - 41	6.29e - 81	2.09e - 64
f_2	1.2	8. 40e - 48	8.59e - 47	6.01e - 49	2.69e - 84	1.28e - 91
	2	9.11e - 33	1.58e - 32	4.31e - 32	3.82e - 56	8.29e - 58
f_3	0	1.59e - 100	8.67e - 115	4.04e - 115	1.83e - 100	2.50e - 100
	1	6.92e - 95	4.74e - 57	3.93e - 57	1.78e - 52	2.25e - 52
f_4	0.5	1.57e - 78	2.54e - 78	1.05e - 78	3.55e - 102	6.54e - 101
	1	1.80e - 83	5.05e - 83	5.94e - 83	6.49e - 93	4.81e - 93
f_5	1.8	9.55e - 42	4.64e - 40	4.88e - 43	1.29e - 69	5.63e - 76
	2.5	1.29e - 28	1.59e - 30	1.68e - 29	1.69e - 64	3.29e - 39
f_6	2	2.59e - 72	1.40e - 81	3.18e - 82	1.07e - 109	3.94e - 101
	2.5	3.53e - 54	4.41e - 61	2.20e - 60	1.13e -89	1.57e - 83
f_7	-1	8.63e - 33	2.25e - 39	1.32e - 43	1.08e - 47	4.96e - 50
	-1.3	2.47e - 56	3.46e - 69	5.76e - 68	1.47e - 70	7.00e - 68
f_8	$.5 \pm 1.5i$	6.89e - 20	1.33e - 20	1.68e - 19	3.86e - 46	1.05e - 23

6. Conclusions

In this work, we have presented a simple, and powerful iterative scheme, which can be used for constructing various one-point third order iterative methods for solving nonlinear equations. The scheme is based on introducing a weight factor in the classical Newton method. The convergence properties of this scheme make it very useful and interesting since all well-known methods, which require one evaluation of each of f, f' and f'' per iteration, satisfy these properties. Using this technique we have developed some new families of third order methods and the existing methods have been shown as special members of the families. The numerical experimentations displayed in Tables 2 and 3 overwhelmingly support the theoretical results.

Table 3: Continuation of error for the same total number of function evaluations.

$f(x)$	x_0	$\lvert e_n \rvert = \lvert x_n - r \rvert$				
		HM	OM	M1	M2	M3
f_1	1	1.35e - 61	3.82e - 83	1.16e - 57	5.98e - 166	3.43e - 62
	2	2.82e - 53	2.22e - 69	1.72e - 67	6.83e - 155	1.58e - 96
f_2	1.2	6. 25e - 65	6.34e - 85	2.53e - 67	2.97e - 153	7.33e - 81
	2	3.47e - 39	2.85e - 46	1.43e - 58	1.13e - 54	3.22e - 47
f_3	0	3.52e - 106	4.63e - 103	6.00e - 101	1.18e - 101	5.56e - 103
	1	1.10e - 54	1.52e - 53	8.89e - 53	5.57e - 53	1.70e - 53
f_4	0.5	7.50e - 87	9.01e - 93	6.90e - 113	1.93e - 96	2.03e - 92
	1	4.42e - 87	1.04e - 89	1.77e - 92	2.89e - 91	9.16e - 90
f_5	1.8	5.76e - 61	3.25e - 89	6.63e - 52	9.10e - 97	7.20e - 82
	2.5	6.66e - 41	1.46e - 58	4.48e - 51	5.25e - 76	4.56e - 97
f_6	2	6.75e - 99	2.59e - 124	1.60e - 101	5.53e - 135	8.19e - 122
	2.5	1.17e - 74	3.21e - 96	1.57e - 97	1.23e -110	4.73e - 102
f_7	-1	4.22e - 92	1.62e - 61	3.64e - 36	1.41e - 54	2.45e - 64
	-1.3	4.76e - 104	2.00e - 85	2.72e - 79	3.73e - 75	1.40e - 82
f_8	$.5 \pm 1.5i$	2.04e - 29	1.11e - 45	1.95e - 39	1.15e - 52	5.12e - 35

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