Journal of Numerical Mathematics and Stochastics, **4** (1) : 1-12, 2012 © ***JNM@S***
http://www.jnmas.org/jnmas4-1.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

A Note on the Existence and Uniqueness of Mild Solutions to Neutral Stochastic Partial Functional Integrodifferential Equations With Non-Liphschitz Coefficients

M. A. DIOP[1], **K. EZZINBI**[2], **and M. LO**[1]

[1]UFR SAT, Département de Mathématiques, Université Gaston Berger de Saint-Louis, BP 234, Saint-Louis, Sénégal; [2]Département de Mathématiques, Université Cadi Ayyad, Faculté des Sciences Semlalia, BP 2390, Marrakech, Maroc, E-mail: mamadou-abdoul.diop@ugb.edu.sn

Abstract. *The article presents results on existence and uniqueness of mild solutions to some neutral stochastic partial functional integrodifferential equations under Carathéodory-type conditions. The results are obtained by using the method of Picard approximation and generalize the results that were reported by Bao and Hou in [3]. The theory of resolvent operators, developed in [2], is employed to demonstrate the existence of these mild solutions. A practical example is provided to illustrate the viability of the abstract results of this work.*

Key words : Resolvent Operators, C_0-Semigroup, Neutral Stochastic Partial Functional Integrodifferential Equations, Wiener Process, Picard Iteration, Mild Solutions.

AMS Subject Classifications : 93E15, 60H15, 35R12

1. Introduction

Neutral stochastic partial functional differential equations arise in many areas of applied mathematics. For this reason, the study of this type of equations has been receiving increased attention in the last few years (see, e.g. [8], [3], [1], [9], [10], [7] and references therein). In [3], Bao and Hou studied, in particular, a stochastic neutral partial functional differential equation. Our intention in this work is to extend these results to stochastic neutral partial functional integrodifferential equations. Our work can in fact be regarded as extension and further development of the work in ([4],[3]).

The present analysis focuses on the following neutral stochastic partial functional integrodifferential equation in a real separable Hilbert space.

$$\begin{cases} d\,[u(t)+G(t,u_t)] = A[u(t)+G(t,u_t)]dt & + \left[\int_0^t B(t-s)[u(s)+G(s,u_s)]ds + F(t,u_t)\right]dt \\ & + H(t,u_t)dw(t),\ for\ t\in[0,T], \\ u_0(.) = \varphi \in C = C([-r,0];\mathbb{H}) & ,\quad where\ \ r>0. \end{cases} \tag{1}$$

Here $u_t(\theta) = u(t+\theta)$ *for* $\theta \in [-r,0]$. The mappings
$G : \mathbb{R}_+ \times C \to \mathbb{H}$, $F : \mathbb{R}_+ \times C \to \mathbb{H}$, *and* $H : \mathbb{R}_+ \times C \to \mathcal{L}(\mathbb{K},\mathbb{H})$ are borel measurable.

The aim of this paper is to study the solvability of (1) and to investigate the existence of mild solution to (1) relying on the Picard iteration. An example is moreover presented to illustrate the applicability of the obtained abstracts results. These results rely essentially on techniques employing a strongly continuous family of operators $\{R(t) : t \geq 0\}$, defined on a Hilbert space $\mathbb{H}$, and utilize their resolvent (the precise definition of this will be given below). Beyond existence and uniqueness, one should also investigate the qualitative effects of pertaining solutions. These effects will be the topic of forthcoming works.

In Section 2 we will firstly introduce some essential notations, concepts and basic results about the Wiener process and deterministic integrodifferential equations. The existence and uniqueness of mild solutions are studied in Section 3 by means of the Picard iteration. Finally, in Section 4 we apply the obtained results, pertaining to (1), to illustrate their applicability.

2. Wiener Process and Deterministic Integrodifferential Equations

2.1. **Wiener process**

Throughout this paper, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Moreover, let $\mathbb{H}$ and $\mathbb{K}$ be two real separable Hilbert spaces; we denote by $\langle .,.\rangle_{\mathbb{H}}$, $\langle .,.\rangle_{\mathbb{K}}$ their inner products and by $\|.\|_{\mathbb{H}}$, $\|.\|_{\mathbb{K}}$ their vector norms, respectively. We denote by $\mathcal{L}(\mathbb{H},\mathbb{K})$ the space of all bounded linear operators from $\mathbb{H}$ into $\mathbb{K}$, equipped with the usual operator norm $\|.\|$. Throughout this paper, when no confusion possibly arises, we shall always use the same symbol $\|.\|$ to denote norms of operators regardless of the spaces potentially involved. Let $r>0$ and $C = C([-r,0];\mathbb{H})$ denote the family of all continuous $\mathbb{H}$-valued functions ξ from $[-r,0]$ to $\mathbb{H}$ with norm $\|\xi\|_C = \sup_{t\in[-r,0]} \|\xi(t)\|_{\mathbb{H}}$. And let $C^b_{\mathcal{F}_0}([-r,0];\mathbb{H})$ be the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $C([-r,0];\mathbb{H})$-valued random variables.

Let $\{w(t) : t \geq 0\}$ denote a $\mathbb{K}$-valued Wiener process defined on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with covariance operator Q; that is, $E\langle w(t),x\rangle_{\mathbb{K}}\langle w(s),y\rangle_{\mathbb{K}} = (t\wedge s)\langle Qx,y\rangle_{\mathbb{K}}$, for all $x,y\in\mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular, we denote by w(t) a $\mathbb{K}$-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. To define stochastic integrals with respect to the Q-Wiener process $w(t)$, we introduce the subspace $\mathbb{K}_0 = Q^{1/2}\mathbb{K}$ of $\mathbb{K}$, endowed with the inner product $\langle u,v\rangle_{\mathbb{K}_0} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_{\mathbb{K}}$ as a Hilbert space. We assume further that there exists a complete orthonormal system $\{e_i\}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i, i = 1, 2, \dots$, and a sequence $\{\beta_i(t)\}_{i>1}$ of independent standard Brownian motions such that

$$w(t) = \sum_{i=1}^{+\infty} \sqrt{\lambda_i}\, \beta_i(t) e_i, \quad t \geq 0,$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Finally, assume $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ to be the space of all Hilbert—Schmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$. It turns out to be a separable Hilbert space equipped with the norm $\|v\|_{\mathcal{L}_2^0} = tr((vQ^{1/2})(vQ^{1/2})^*)$ for any $v \in \mathcal{L}_2^0$. Obviously, for any bounded operator $v \in \mathcal{L}_2^0$, this norm reduces to $\|v\|_{\mathcal{L}_2^0}^2 = tr(vQv^*)$.

2.2. **Partial integrodifferential equations**

In this subsection, we recall some fundamental results needed to establish our results. As for the theory of resolvent operators, we refer the reader to [2, 6] . Throughout this paper, X is a Banach space, A and $B(t)$ are closed linear operators on X. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \qquad for \quad y \in Y.$$

The notations $C([0, +\infty); Y), B(Y, X)$ stand for the space of all continuous functions from $[0, +\infty)$ into Y, the set of all bounded linear operators from Y into X, respectively. We are able then to invoke the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds \quad for \quad t \geq 0, \\ v(0) = v_0 \in X. \end{cases} \tag{2}$$

Definition 2.1.[2] A resolvent operator for Eq.(2) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$, having the following properties:

- (i) $R(0) = I$ and $|R(t)| \leq Ne^{\beta t}$, for some constants N and β.
- (ii) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.
- (iii) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y, R(.)x \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad for \quad t \geq 0. \end{aligned}$$

In what follows we make the following assumptions :
(**H1**): A is the infinitesimal generator of a strongly continuous semigroup on X.
(**H2**): For all $t \geq 0$, $B(t)$ is a closed linear operator from $D(A)$ to X, and $B(t) \in B(Y, X)$. For any $y \in Y$, the map $t \to B(t)y$ is bounded differentiable and the derivative $t \to B'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

The resolvent operator plays an important role in the study of the existence of solutions and in providing a variation-of-constants formula for nonlinear systems. We, however, need to know when the linear system (2) has a resolvent operator. For more details on resolvent operators, we refer the reader to [2, 6]. In actual fact, the following theorem gives a satisfactory answer to this problem, and it will be used in this work to develop our main results.

Theorem 2.1.[2] *Assume that (H1)-(H2) hold. Then there exists a unique resolvent operator of the Cauchy problem(2).*

Let us now give some results on the existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t), \quad \textit{for} \quad t \geq 0, \\ v(0) = v_0 \in X, \end{cases} \tag{3}$$

where $q : [0,+\infty[\to X$ is a continuous function.

Definition 2.2.[2] A continuous function $v : [0,+\infty) \to X$ is said to be a strict solution of Eq.(3) if
(i) $v \in C^1([0,+\infty);X) \cap C([0,+\infty);Y)$,
(ii) v satisfies Eq.(3) for $t \geq 0$.

Remark 2.1. From this definition, we deduce that $v(t) \in D(A)$, the function $B(t-s)v(s)$ is integrable, for all $t \geq 0$, and $s \in [0,t]$.

Theorem 2.2.[2] *Assume that (H1)-(H2) hold . If v is a strict solution of Eq.(3), then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds, \qquad \textit{for} \quad t \geq 0. \tag{4}$$

Accordingly, we we are able to state the following definition.

Definition 2.3.[2] For $v_0 \in X$, a function $v : [0,+\infty) \to X$ is called a mild solution of (3) if v satisfies (4).

The next theorem provides sufficient conditions for the regularity of solutions of Eq. (3).

Theorem 2.3.[2] *Let $q \in C^1([0,+\infty);X)$ and v be defined by (4). If $v_0 \in D(A)$, then v is a strict solution of Eq.(3).*

For convenience, we invoke from [5] the mild solution to (1) as follows.

Definition 2.4. A process $\{u(t),\ 0 \leq t \leq T\}$, $0 \leq T < +\infty$, is called a mild solution of Eq.(1) if
(i) $u(t)$ is $\mathcal{F}_t$ –adapted and continuous for $t \geq 0$, almost surely,
(ii) For arbitrary $t \in [0,T]$, $P\{\omega \ : \int_0^t \|u(s)\|_{\mathbb{H}}^2 ds < +\infty\} = 1$ and almost surely

$$u(t) + G(t,u_t) = R(t)[\varphi(0) + G(0,\varphi)] + \int_0^t R(t-s)F(s,u_s)ds + \int_0^t R(t-s)H(s,u_s)dw(s). \tag{5}$$

To guarantee the existence and uniqueness of a mild solution to Eq.(1) the following much

weaker conditions (instead of the global Lipschitz condition and linear growth) are listed.
(**H3**): The mappings $F(.)$ and $H(.)$ satisfy, for any $\zeta,\eta\in\mathbb{H}$ and $t\geq 0$, the following non-Lipschitz condition:

$$\|F(t,\zeta)-F(t,\eta)\|_{\mathbb{H}}^2+\|H(t,\zeta)-H(t,\eta)\|_{\mathcal{L}_2^0}^2\leq\lambda(\|\zeta-\eta\|_C^2),$$

where $\lambda(.)$ is a concave nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\lambda(0)=0$, $\lambda(u)>0$, for $u>0$ and $\int_{0^+}\frac{du}{\lambda(u)}=+\infty$, e.g., $\lambda(u)\sim u^\alpha$, $\frac{1}{2}<\alpha<1$.
(**H4**): There is an $M>0$ such that

$$\sup_{0\leq t\leq T}(\|F(t,0)\|_{\mathbb{H}}^2\vee\|H(t,0)\|_{\mathcal{L}_2^0}^2)\leq M.$$

(**H5**): The mapping $G(t,x)$ satisfies, when there exists $K>0$, such that for any $\zeta,\eta\in\mathbb{H}$ and $t\geq 0$,

$$\|G(t,\zeta)-G(t,\eta)\|_{\mathbb{H}}\leq K\|\zeta-\eta\|_C.$$

To develop our main results we shall need in the sequel the following lemmas.

Lemma 2.1. ([12], theorem 1.8.2, p. 45) *Let $a>0$, $c>0$ and $\kappa:R^+\to R^+$ be a continuous nondecreasing function, such that $\kappa(t)>0$ for all $t>0$. Let u(.) be a Borel measurable bounded nonnegative function on [0, a], and assume v(.) to be a nonnegative integrable function on [0, T]. If*

$$u(t)\leq c+\int_0^t v(s)\kappa(u(s))ds,\quad\forall\ t\in[0,a],$$

then

$$u(t)\leq J^{-1}\left(J(c)+\int_0^t v(s)ds\right),$$

holds for all $t\in[0,a]$ such that $J(c)+\int_0^t v(s)ds\in Dom(J^{-1})$, where $J(\tau)=\int_0^\tau\frac{ds}{\kappa(s)}$, on $\tau>0$, and J^{-1} is the inverse function of J. In particular, if, moreover, $c=0$ and $\int_{0^+}\frac{ds}{\kappa(s)}=+\infty$, then $u(t)=0$ for all $t\in[0,a]$.

Lemma 2.2. ([11], lemma1) *For $x,y\in H$ and $0<c<1$, the following inequality is true.*

$$\|x\|_{\mathbb{H}}^2\leq\frac{1}{1-c}\|x-y\|_{\mathbb{H}}^2+\frac{1}{c}\|y\|_{\mathbb{H}}^2.$$

3. Existence and Uniqueness

In this section, we move to study the existence and uniqueness of mild solutions to neutral stochastic partial functional integrodifferential equations under a non-Lipschitz condition and a weakened linear growth condition. To complete our main results, we need to develop several lemmas which will be utilized in the sequel.

Invoke the following Picard iteration which is defined by

$$u^0(t)=R(t)\varphi(0)\quad for\ t\in[0,T],$$

$$u^0(t)=\varphi(t)\quad for\ t\in[-r,0],$$

and u^n for $n\geq 1$ is defined by

$$u^n(t)=\varphi(t)\quad for\quad t\in[-r,0],$$

and

$$u^n(t) + G(t,u_t^n) = R(t)[\varphi(0) + G(0,\varphi)] + \int_0^t R(t-s)F(s,u_s^{n-1})ds$$
$$+ \int_0^t R(t-s)H(s,u_s^{n-1})dw(s),\ t \in [0,T]. \quad (6)$$

Lemma 3.1. *Let the hypotheses (H1)-(H5) hold and* $K < 1$. *Then there is a positive constant* $\tilde{C}$, *which is independent of* $n \geq 1$, *such that for any* $t \in [0,T]$,

$$E \sup_{0\leq t\leq T} \|u^n(t)\|_{\mathbb{H}}^2 \leq \tilde{C}. \quad (7)$$

Proof. For $0 \leq t \leq T$, it follows easily from (6) that

$$\begin{aligned} E \sup_{0\leq t\leq T} \|u^n(t) + G(t,u_t^n)\|_{\mathbb{H}}^2 &\leq 3E \sup_{0\leq s\leq t} \|R(t)[\varphi(0) + G(0,\varphi)]\|_{\mathbb{H}}^2 \\ &+3E \sup_{0\leq t\leq T} \left\| \int_0^t R(t-s)F(s,u_s^{n-1})ds \right\|_{\mathbb{H}}^2 \\ &+3E \sup_{0\leq t\leq T} \left\| \int_0^t R(t-s)H(s,u_s^{n-1})dw(s) \right\|_{\mathbb{H}}^2 \\ &=: 3(I_1 + I_2 + I_3). \end{aligned} \quad (8)$$

Employing the assumption $(H5)$ results with

$$I_1 \leq M_1(1+K)^2 E\|\varphi\|_C^2, \quad (9)$$

where

$$M_1 = \sup_{0\leq t\leq T} \|R(t)\|^2.$$

On another note, in view of $(H4)$, we deduce from Hölder's inequality that

$$\begin{aligned} I_2 &\leq T\, E \sup_{0\leq t\leq T} \int_0^t \|R(t-s)[F(s,u_s^{n-1}) - F(s,0) + F(s,0)]\|_{\mathbb{H}}^2 ds \\ &\leq 2TM_1\Big[MT + \int_0^T E\lambda(\|u_s^{n-1}\|_C^2)ds\Big]. \end{aligned} \quad (10)$$

Next, according to Liu ([5], Theorem 1.2.6, p. 14]) together with $(H4)$, there exists a constant $C > 0$ such that

$$\begin{aligned} I_3 &\leq C\int_0^T \|[H(s,u_s^{n-1}) - H(s,0) + H(s,0)]\|_{\mathcal{L}_2^0} ds \\ &\leq 2C\Big[MT + \int_0^t E\lambda(\|u_s^{n-1}\|_C^2)ds\Big]. \end{aligned} \quad (11)$$

Since $\lambda(u)$ is concave on $u \geq 0$, then there is a pair of positive constants a, b such that

$$\lambda(u) \leq a + bu.$$

Putting (9)-(11) into (8) yields, for some positive constants C_1 and C_2, that

$$E\sup_{0\le t\le T}\|u^n(t)+G(t,u_t^n)\|_{\mathbb{H}}^2 \le C_1 + C_2\int_0^T E\|u_s^{n-1}\|_C^2 ds. \tag{12}$$

While, by Lemma 2.2 for $K<1$, it follows that

$$E(\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2) \le \frac{1}{1-K}E\sup_{0\le t\le T}\|u^n(t)+G(t,u_t^n)\|_{\mathbb{H}}^2 + \frac{1}{K}E\sup_{0\le t\le T}\|G(t,u_t^n)\|_{\mathbb{H}}^2$$

$$\le \frac{1}{1-K}E\sup_{0\le t\le T}\|u^n(t)+G(t,u_t^n)\|_{\mathbb{H}}^2 + K\,E(\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2) + K\,E\|\varphi\|_C^2,$$

which further implies that

$$E(\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2) \le \frac{1}{(1-K)^2}E\sup_{0\le t\le T}\|u^n(t)+G(t,u_t^n)\|_{\mathbb{H}}^2 + \frac{K}{1-K}E\|\varphi\|_C^2.$$

Thus, by (12) we have

$$E(\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2) \le \frac{C_1}{(1-K)^2} + \left[\frac{C_2T}{(1-K)^2} + \frac{K}{1-K}\right]E\|\varphi\|_C^2 + \frac{2C_2}{1-K}\int_0^T \sup_{0\le\theta\le s}\|u^{n-1}(\theta)\|_{\mathbb{H}}^2 ds.$$

Observing that

$$\max_{1\le n\le k} E\sup_{0\le t\le T}\|u^{n-1}(t)\|_{\mathbb{H}}^2 \le E\|\varphi\|_C^2 + \max_{1\le n\le k} E\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2,$$

allows, for some positive constants C_3 , C_4, to write

$$\max_{1\le n\le k} E\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2 \le C_3 + C_4\,E\int_0^T \max_{1\le n\le k} E\sup_{0\le\theta\le s}\|u^n(\theta)\|_{\mathbb{H}}^2 ds.$$

Now, an application of the well-known Gronwall's inequality yields that

$$\max_{1\le n\le k} E\sup_{0\le t\le T}\|u^n(t)\|_{\mathbb{H}}^2 \le C_3 + \exp(C_4T).$$

Since k is arbitrary, the required assertion (7) directly follows. ■

Lemma 3.2. *Let the conditions* $(H1)-(H4)$ *be satisfied. We further assume that*

$$K<1. \tag{13}$$

Then there exists a positive constant $\tilde{K}$ *such that, for all* $0\le t\le T$ *and* $n,m\ge 1$,

$$E\sup_{0\le s\le t}\|u^{n+m}(s)-u^n(s)\|_{\mathbb{H}}^2 \le \tilde{K}\int_0^t \lambda\left(E\sup_{0\le l\le s}\|u^{n+m-1}(s)-u^{n-1}(s)\|_{\mathbb{H}}^2\right)ds. \tag{14}$$

Proof. From (6) it is easy to see that, for any $0\le t\le T$,

$$E\sup_{0\le s\le t}\|u^{n+m}(s)-u^n(s)+G(s,u_s^{n+m}(s)) - G(s,u_s^n(s))\|_{\mathbb{H}}^2$$

$$\le 2E\sup_{0\le s\le t}\left\|\int_0^s R(s-l)[F(l,u_l^{n+m-1})-F(l,u_l^{n-1})]dl\right\|_{\mathbb{H}}^2$$

$$+2E\sup_{0\le s\le t}\left\|\int_0^s R(s-l)[H(l,u_l^{n+m-1})-H(l,u_l^{n-1})]dw(l)\right\|_{\mathbb{H}}^2 =: J_1+J_2.$$

The proof of Lemma 3.1 indicates the existence of a positive constant C_5 satisfying

$$J_1 + J_2 \leq C_5 \int_0^t \lambda\left(E \sup_{0\leq l\leq s} \|u^{n+m-1}(l) - u^{n-1}(l)\|^2_{\mathbb{H}}\right)ds.$$

Moreover, Lemma 3.1 and $(H5)$ imply that

$$E \sup_{0\leq s\leq t} \|u^{n+m}(s) - u^n(s)\|^2_{\mathbb{H}} \leq \frac{1}{1-K} E \sup_{0\leq s\leq t} \|u^{n+m}(s) - u^n(s) + G(s,u_s^{n+m}(s)) - G(s,u_s^n(s))\|^2_{\mathbb{H}}$$
$$+ K E \sup_{0\leq s\leq t} \|u^{n+m}(s) - u^n(s)\|^2_{\mathbb{H}}.$$
$$\leq \frac{C_5}{1-K} \int_0^t \lambda\left(E \sup_{0\leq l\leq s} \|u^{n+m-1}(l) - u^{n-1}(l)\|^2_{\mathbb{H}}\right)ds + K E \sup_{0\leq s\leq t} \|u^{n+m}(s) - u^n(s)\|^2_{\mathbb{H}}.$$

So the desired assertion (14) follows from the validity of (13). ■

It is possible now to state our main result.

Theorem 3.1. *Under the conditions of Lemma 3.2, then Eq.(1) admits a unique mild solution.*

Proof. *Uniqueness*: Denote by $u_1(t)$ and $u_2(t)$ two mild solutions to (1). In the same way as Lemma 3.2 was proved, we can show that, for some $D > 0$,

$$E \sup_{0\leq s\leq t} \|u_1(s) - u_2(s)\|^2_{\mathbb{H}} \leq D\int_0^t \lambda\left(E \sup_{0\leq l\leq s} \|u_1(l) - u_2(l)\|^2_{\mathbb{H}}\right)ds.$$

This, together with Lemma 3.2, leads to

$$E \sup_{0\leq s\leq t} \|u_1(s) - u_2(s)\|^2_{\mathbb{H}} = 0,$$

which further implies that $u_1(s) = u_2(s)$ almost surely for any $0 < t \leq T$.
Existence: Following also the proof of Lemma 3.2, there exists a positive constant $\overline{C}$ such that, for all $0 \leq t \leq T$ and $n, m \geq 1$,

$$E \sup_{0\leq s\leq t} \|u^{n+1}(s) - u^{m+1}(s)\|^2_{\mathbb{H}} \leq \overline{C}\int_0^t \lambda\left(E \sup_{0\leq l\leq s} \|u^n(s) - u^m(s)\|^2_{\mathbb{H}}\right)ds.$$

Integrating both sides and applying the Jensen's inequality gives

$$\int_0^t E \sup_{0\leq l\leq s} \|u^{n+1}(l) - u^{m+1}(l)\|^2_{\mathbb{H}} ds \leq \overline{C}\int_0^t\int_0^s \lambda\left(E \sup_{0\leq l\leq s} \|u^n(l) - u^m(l)\|^2_{\mathbb{H}}\right)dlds$$
$$= \overline{C}\int_0^t s\int_0^s \lambda\left(E \sup_{0\leq l\leq s} \|u^n(l) - u^m(l)\|^2_{\mathbb{H}}\right)\frac{1}{s}dlds$$
$$\leq \overline{C}t\int_0^t \lambda\left(\int_0^s E \sup_{0\leq l\leq s} \|u^n(l) - u^m(l)\|^2_{\mathbb{H}}\frac{1}{s}dl\right)ds.$$

Then

$$v_{n+1,m+1}(t) \leq \overline{C}\int_0^t \lambda(v_{n,m}(s))ds,$$

where

$$v_{n,m}(t) = \frac{1}{t}\int_0^t E \sup_{0\leq l\leq s} \|u^n(l) - u^m(l)\|^2_{\mathbb{H}} ds.$$

While by Lemma 3.1, it is easy to see that

$$\sup_{n,m\to\infty} v_{n,m}(t) < \infty.$$

So letting $v(t) := \lim \sup_{n,m\to\infty} v_{n,m}(t)$ and invoking the Fatou's lemma yields

$$v(t) \leq \overline{C}\int_0^t \lambda(v(s))ds.$$

Next, apply Lemma 3.2 to realize immediately, for any $t \in [0,T]$, that $v(t) = 0$. This further means that $\{u^n(t), \ : n \in \mathbb{N}\}$ is a Cauchy sequence in L^2. So there is a $u \in L^2$ satisfying

$$\lim_{n\to\infty} \int_0^T E \sup_{0\leq s\leq t} \|u^n(s) - u(s)\|^2_{\mathbb{H}} = 0.$$

Moreover, by Lemma 3.2, it is easy to conclude that $E\|u(t)\|^2_{\mathbb{H}} \leq C$. Hence in what follows we claim that $u(t)$ is a mild solution to Eq.(1). Indeed, on one hand, by $(H4)$, the Hölder's inequality, according Liu ([5], Theorem 1.2.6, p. 14) and letting $n \to \infty$, for $0 \leq t \leq T$, we can also claim, for $t \in [0,T]$, that

$$\left\| \int_0^t R(t-s)[F(s,u_s^{n-1}) - F(s,u_s)]ds \right\|^2_{\mathbb{H}} \to 0, \quad E\left\| \int_0^t R(t-s)[H(s,u_s^{n-1}) - H(s,u_s)]ds \right\|^2_{\mathbb{H}} \to 0.$$

On the other hand, by applying $(H5)$, we can also claim, for $t \in [0,T]$, that

$$E\|G(s,u_s^n) - G(s,u_s)\|^2_{\mathbb{H}} \leq K^2 E \sup_{0\leq s\leq t} \|u^n(s) - u(s)\|^2_{\mathbb{H}} \to 0.$$

Now taking limits in both sides of (6) leads, for $t \geq 0$, to

$$u(t) = R(t)[\varphi(0) + G(0,\varphi)] - G(t,u_t) + \int_0^t R(t-s)F(s,u_s)ds + \int_0^t R(t-s)H(s,u_s)dw(s).$$

This is an illustration that u is a mild solution to of Eq.(1) on [0, T]. ■

4. Application

We conclude this work with the example

$$\begin{cases} \frac{\partial}{\partial t}\left[x(t,\xi) + \int_{-r}^0 g(t,x(t+\theta,\xi))d\theta\right] = \frac{\partial^2}{\partial\xi^2}\left[x(t,\xi) + \int_{-r}^0 g(t,x(t+\theta,\xi))d\theta\right] \\ +\int_0^t b(t-s)\frac{\partial^2}{\partial\xi^2}\left[x(s,\xi) + \int_{-r}^0 g(t,x(t+\theta,\xi))d\theta\right]ds \\ +\int_{-r}^0 f(t,x(t+\theta,\xi))d\theta + h(t,x(t+\theta,\xi))dw(t) \quad for\ t \geq 0\ \ and\ \xi \in [0,\pi] \\ x(t,0) + \int_{-r}^0 g(t,x(t+\theta,0))d\theta = 0\ \ for\ t \geq 0 \\ x(t,\pi) + \int_{-r}^0 g(t,x(t+\theta,\pi))d\theta = 0\ \ for\ t \geq 0 \\ x(\theta,\xi) = x_0(\theta,\xi)\ \ for\ \theta \in [-r,0]\ \ and\ \ \xi \in [0,\pi], \end{cases} \tag{15}$$

where $w(t)$ denotes a $\mathbb{R}$-valued Brownian motion, $g, f, h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions , $b : \mathbb{R}^+ \to \mathbb{R}$ is continuous and $x_0 : [-r, 0] \times [0, \pi] \to \mathbb{R}$ is a given continuous function such that $x_0(.) \in L^2([0, \pi])$ is $\mathcal{F}_0$-measurable and satisfies $E\|x_0\|^2 < \infty$.
Let $\mathbb{H} = L^2([0, \pi])$ with the norm $\|.\|$ and $e_n := \sqrt{\frac{2}{\pi}} \sin(nx)$, $(n = 1, 2, 3, \cdots)$ denote the complete orthonormal basis in $\mathbb{H}$. Also let $w(t) := \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \beta_n(t) e_n$ $(\lambda_n > 0)$, where $\beta_n(t)$ are one-dimensional standard Brownian motions mutually independent on a usual complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Define then $A : \mathbb{H} \to \mathbb{H}$ where $A = \frac{\partial^2}{\partial z^2}$, with the domain $D(A) = H^2([0, \pi]) \cap H_0^1([0, \pi])$.
Consequently $Ah = -\sum_{n=1}^{\infty} n^2 < h, e_n > e_n$, $h \in D(A)$, where e_n, $n = 1, 2, 3, \cdots$, is also the orthonormal set of eigenvectors of A.

It is well-known that A is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{H}$, thus (H1) is true. Furthermore let $B : D(A) \subset \mathbb{H} \to \mathbb{H}$, be the operator defined by $B(t)(z) = b(t)Az$ for $t \geq 0$ and $z \in D(A)$. We may suppose then that

(i) There exists a positive constant L_g, $r\sqrt{L_g} \geq 0$, such that

$|g(t, \zeta_1) - g(t, \zeta_2)|^2 \leq L_g |\zeta_1 - \zeta_2|^2$.

(ii) There exists a constant L_f, $0 < \pi r^2 L_f < 1$, such that

$|f(t, \zeta_1) - f(t, \zeta_2)|^2 \leq L_f\, \lambda(\|\zeta_1 - \zeta_2\|_C^2)$.

(iii) There exists a constant L_h, $0 < \pi r L_h < 1$, such that

$|h(t, \zeta_1) - h(t, \zeta_2)|^2 \leq \lambda(\|\zeta_1 - \zeta_2\|_C^2)$.

Also let $C = C([-r, 0]; \mathbb{H})$ and define the operators $G, F, H : \mathbb{R}^+ \times C \to \mathbb{H}$ by
$G(t, \phi)(\xi) = \int_{-r}^{0} g(t, \phi(\theta)(\xi)) d\theta$ *for* $\xi \in [0, \pi]$ *and* $\phi \in C$,
$F(t, \phi)(\xi) = \int_{-r}^{0} f(t, \phi(\theta)(\xi)) d\theta$ *for* $\xi \in [0, \pi]$ *and* $\phi \in C$,
$H(t, \phi)(\xi) = h(t, \phi(\theta)(\xi))$ *for* $\xi \in [0, \pi]$ *and* $\phi \in C$.
Now if we put

$$\begin{cases} u(t) = x(t, \xi) \text{ for } t \geq 0 \text{ and } \xi \in [0, \pi] \\ \varphi(\theta)(\xi) = x_0(\theta, \xi) \text{ for } \theta \in [-r, 0] \text{ and } \xi \in [0, \pi]. \end{cases}$$

Then Eq. (15) takes the following abstract form

$$\begin{cases} d\,[u(t) + G(t, u_t)] = A[u(t) + G(t, u_t)]dt + \left[\int_0^t B(t-s)[u(s) + G(s, u_s)]ds + F(t, u_t)\right]dt \\ \qquad\qquad + H(t, u_t)dw(t), \text{ for } t \in [0, T], \\ u_0(.) = \varphi \in C = C([-r, 0]; \mathbb{H}) \quad , \text{ where } r > 0. \end{cases} \tag{16}$$

Moreover, if b is bounded and C^1 function such that b' is bounded and uniformly continuous, then $(H1)$ and $(H2)$ are satisfied and hence, by Theorem 2.2, Eq. (2) has a resolvent operator $(R(t))_{t\geq 0}$ on $\mathbb{H}$. The assumption (i) implies moreover that

$\|G(t, \phi_1) - G(t, \phi_2)\|_{L^2[0, \pi]} \leq r\sqrt{L_g}\ \lambda(\|\phi_1 - \phi_2\|_C^2)$.

And by assumptions (ii) and (iii) we have

$$\|F(t,\phi_1) - F(t,\phi_2)\|_{L^2[0,\,\pi]} \le r^2\pi\, L_f\, \lambda(\|\phi_1 - \phi_2\|_C^2).$$

$$\|H(t,\phi_1) - H(t,\phi_2)\|_{L^2[0,\,\pi]} \le r\pi\, L_h\, \lambda(\|\phi_1 - \phi_2\|_C^2).$$

Thus, all the stipulations of Theorem 3.3 are fulfilled, and the existence of a unique mild solution of Eq.(15) has been demonstrated.

Acknowledgements

The authors wish to thank an anonymous referee for his critical reading of an earlier version of this paper.

References

[1] T. E. Govindan, Almost sure exponential stability for stochastic neutral partial functional differential equations, *Stochastic* **77**, (2005), 139-154.

[2] R. Grimmer, Resolvent operators for integral equations in a Banach space, *Transactions of the American Mathematical Society* **273**, (1982), 333-349.

[3] J. Bao, and Z. Hou, Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients, *Computers and Mathematics with Applications* **59**, (2010), 207-214.

[4] M. A. Diop, K. Ezzinbi, and M. Lo, Existence and uniqueness of mild solutions for some neutral stochastic partial functional integrodifferential equations with non-Lipschitz coefficients, *International Journal of Mathematics and Mathematical Sciences* **2012**, (2012), Article ID748590, 12 pages, doi:10.1155/2012/748590.

[5] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, Chapman and Hall, CRC, London, 2006.

[6] R. Grimmer, and A. J. Pritchard, Analytic resolvent operators for integral equations, *Journal of Differential Equations* **50**, (1983), 234-259.

[7] F. Jiang, and Y. Shen, A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients, *Computers and Mathematics with Applications* **61**, (2011), 1590-1594.

[8] T. Taniguchi, Successive approximations of solutions of stochastic differential equations, *Journal of Differential Equations* **96**, (1992), 152-169.

[9] J. Turo, Successive approximations of solutions of stochastic differential equations, *Periodica Mathematica Hungarica* **30**, (1995), 87-96.

[10] G. Cao, K. He, and X. Zhang, Successive approximations of infinite dimensional SDES with jump, *Stochastic Systems* **5**, (2005), 609-619.

[11] T. Caraballo, J. Real, and T. Taniguchi, The exponential stability of neutral stochastic delay partial differential equations, *Discrete and Continuous Dynamical Systems* **18**, (2007), 295-313.

[12] G. Da Prato, and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

Article history: Submitted October, 18, 2011; Revised December, 19, 2011; Accepted April, 22, 2012.