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Constrained Backward SDEs With Jumps and American Options

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Abstract. *We consider a class of backward stochastic differential equation (BSDE) driven by a Lévy process subject to constraint on solution, which is not necessary convex. We prove existence and uniqueness of the minimal solution using the Snell envelope and penalization approach. As an application we provide for a minimal wealth to hedge American options and American game options.*

Key words : Backward SDEs, Snell Envelope, Penalization, Portfolio constraint, Reflecting barrier, Mokobodski's hypothesis, American option, Dynkin game.

AMS Subject Classifications : 60H30, 60H10, 60G40

1. Introduction

In this work we are interested in backward stochastic differential equations (BSDE's) under a constraint on the gain-process. Such a BSDE type has been introduced by Cvitanic *et al.* [5]. An unconstrained version of these equations was introduced for the first time by Bismut [2] in the linear case, and was generalized by Pardoux and Peng [15]. Later, Tang and Li [18] considered standard BSDE when the noise is driven by Brownian motion and an independent Poisson random measure. More recently, Barles *et al.* [1] studied the link of those BSDE with viscosity solution of integral-partial differential equations. The topic of BSDE with constraints has been studied by many authors. In particular, one barrier reflected BSDE has been studied by El Karoui *et al.* [7]. In their study, one of the components is forced to stay above a given barrier which is assumed to be a continuous adapted stochastic process. Hu and Tang [13] generalized this result to a multidimensional BSDE with oblique reflection on a convex domain. This problem was motivated by optimal switching problem. An important application of the constrained BSDE is the pricing of the constrained-contingent claim in an incomplete market. Indeed, the problem is to determine the price of a contingent claim $\xi \geq 0$ of maturity

T, which is a contract to pay an amount ξ a time T, where the portfolio of an asset is constrained in a given subset. In this case, (Y,Z) correspond to a solution of a BSDE with generator f which must remain in this subset. Here Y is the replicating portfolio and Z the hedging strategy. In the pricing of an American options under constrained portfolio, the associated BSDE is a reflected BSDE with constrained portfolio, with reflection on Y and constraint on Z.

Cvitanic *et al.* [5], and Buckdahn and Hu [3] studied a case when the portfolio belongs to some convex domain. The generalization to nonconvex situations was studied by Peng [16], where the constraints are in Y and Z. Recently, Kharroubi *et al*. [14] considered the case of constraint on the jump component and the link with quasi-variational inequalities.

The main goal of this paper is to deal with constrained BSDE on portfolio when the noise comes from two sources: a Brownian motion and an independent Poisson random measure. The constraints are assumed to be globally Lipschitz w.r.t their arguments. We will add a reflection on our process Y and provide a financial interpretation by hedging an American contingent claim with constraint on the strategy, as well as the case of two reflected barriers applied to get the value function of an American contingent claim with constraint on the hedging strategy.

This paper is organized as follows. Section 2.1 contains the underlying hypotheses and provides a detailed formulation of the problem. In the case of a generator depending on y and z, we will follow a penalization method as was done in El Karoui et al. [7]. In Section 2.3 we derive an explicit formula of the process Y when the generator does not depend on y and z variables. The pertaining minimal solution is obtained directly by the supermartingale decomposition method in spirit of the work of El Karoui and Quenez [8] or Cvitanic, Karatzas and Soner [5]. An important application of the constrained BSDE is the pricing of contingent claim with constraint of portfolio. This will be done in Section 3.

2. Constrained BSDE

2.1. **Setting of the problem and hypotheses**

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space, on which are defined $(B_t)_{0\leq t\leq T}$ a d-dimensional Brownian motion and Poisson random measure μ on $\mathbb{R}_+\times E$, where E is a compact set of $\mathbb{R}^d$, endowed with its Borel field $\mathcal{E}$. We assume that the Poisson random measure μ is independent of B, and has the intensity measure $\lambda(de)dt$ for some finite measure λ on $(E,\mathcal{E})$. We set $\tilde{\mu}(dt,de)=\mu(dt,de)-\lambda(de)dt$, the compensated measure associated to μ and denote by $\mathbb{F}=(\mathcal{F}_t)_{0\leq t<T}$ the natural filtration generated by B and $\tilde{\mu}$, and by $\mathcal{P}$ the σ-algebra of predictable subsets of $\Omega\times[0,T]$.

For $p\in[1,\infty)$:

• $\mathcal{S}^2$, the set of real-valued càdlàg $\mathbb{F}$-adapted processes $Y=(Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathcal{S}^2}:=\left(\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right]\right)^{\frac{1}{2}}.$$

• $\mathbf{L}^p(0,T)$, $p\geq 1$, the set of real-valued processes $(\phi_t)_{0\leq t\leq T}$ such that

$\mathbb{E}\Big(\Big[\int_0^T|\phi_t|^p dt\Big]\Big)^{\frac{1}{p}} < \infty.$

• $\mathbf{L}^p_{\mathbb{F}}(0,T)$, $p \geq 1$, is the subset of $\mathbf{L}^p(0,T)$ consisting of $\mathbb{F}$-adapted processes.
• $\mathbf{L}^p(B)$, $p \geq 1$, the set of $\mathbb{R}^d$-valued $\mathcal{P}$-measurable processes $Z = (Z_t)_{0\leq t\leq T}$ such that

$$\|Z\|_{L^p(B)} := \left(\mathbb{E}\left[\sup_{0\leq t\leq T}|Z_t|^p\right]\right)^{\frac{1}{p}} < \infty.$$

• $\mathbf{L}^p(\tilde{\mu})$, the set of $\mathcal{P}\bigotimes\mathcal{E}$-measurable maps $V : \Omega\times[0,T]\times E \to \mathbb{R}$ such that

$$\|V\|_{\mathbf{L}^p(\tilde{\mu})} := \Big(\mathbb{E}\Big[\int_0^T\int_E|V_t(e)|^p\lambda(de)dt\Big]\Big)^{\frac{1}{p}}.$$

• $\mathbf{A}^2$ the closed subset of $\mathcal{S}^2$ consisting of nondecreasing processes $K = (K_t)_{0\leq t\leq T}$ with $K_0 = 0$.
• Two obstacles $L := (L_t)_{t\leq T}$ and $U := (U_t)_{t\leq T}$ which are $\mathcal{F}_t$-progressively measurable *rcll*, real-valued processes satisfying $L_t \leq U_t$, $\forall\ t \leq T$ and $L_T \leq \xi \leq U_T$, P-a.s. In addition they belong to $\mathcal{S}^2$, i.e. $\mathbb{E}[\sup_{0\leq t\leq T}(|L_t|^2+|U_t|^2)] < \infty$.

Let us fix any real number x to consider the following forward SDE:

$$X_t = x + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s + \int_0^t\int_E \beta(X_{s^-},e)\tilde{\mu}(ds,de), \tag{1}$$

where $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ are continuous mappings, and $\beta : \mathbf{R}^d\times E \to \mathbf{R}^d$ measurable and satisfying, for some constants k, $K \in \mathbb{R}^+_*$, and for all x, $x' \in \mathbb{R}^d$ and $e \in E$,
• b and σ are continuous in t
• $|b(t,x)-b(t,x')|+|\sigma(t,x)-\sigma(t,x')| \leq k(|x-x'|)$.
• $|\beta(0,e)| \leq K(1\wedge|e|)$.
• $|\beta(x,e)-\beta(x',e)| \leq K|x-x'|(1\wedge|e|)$.

Existence and uniqueness, as well as the properties of the solutions of equation (1) can be found in the work of Fujiwara and Kunita (1989) or Ikeda and Watanabe (1989). The solution $\{X_t, t\geq 0\}$ can be interpreted as the stock price process. Our main goal is to study a generalization of the results obtained in [17] and [3] by considering BSDEs with jumps and assuming unnecessary convex conditions on the constraints.

A portfolio-constrained BSDE with jump, is characterized by three objects:
• A terminal condition, $\xi \in \mathbb{L}^2(\mathcal{F}_T)$.
• A coefficient f, called the generator, which maps $[0,T]\times\Omega\times\mathbb{R}^{1+d}\times L^2(E,\mathcal{E},\lambda,\mathbb{R})$ to $\mathbb{R}$ and is $\mathcal{P}\bigotimes\mathcal{B}(\mathbb{R}^{1+d})\bigotimes\mathcal{B}(L^2(E,\mathcal{E},\lambda,\mathbb{R}))$-measurable satisfying a growth sub-linear condition.
• A constraint function $\phi : [0,T]\times\Omega\times\mathbb{R}^{1+d} \to \mathbb{R}^+$, such that ϕ is globally Lipschitz w.r.t y and z.

Let us focus then on the constrained BSDE with jumps associated with (f,ξ,ϕ).

Definition 2.1. A solution to the portfolio-constrained BSDE with jumps is a quadruple $(Y,Z,V,K) \in \mathcal{S}^2\times L^2(B)\times L^2(\tilde{\mu})\times\mathbf{A}^2$ satisfying:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T\int_E V_s(e)\tilde{\mu}(ds,de), \tag{2}$$

for $0 \leq t \leq T$ a.s.
The process K acts in an optimal way such that the constraint

$$\phi(t, Y_t, Z_t) \geq 0 \quad a.s.,\ a.e. \tag{3}$$

For any quadruple $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K}) \in \mathcal{S}^2 \times L^2(B) \times L^2(\tilde{\mu}) \times \mathbf{A}^2$ satisfying (2)-(3), we have $Y \leq \tilde{Y}\ a.s.$

In this case, we say that (Y, Z, V, K) is the minimal solution. The increasing process K acts directly on Y in such a way that the solution evolves so as $\phi(t, Y_t, Z_t) \geq 0$. Let us now, for the sake of completeness, describe the classical conditions on coefficients for this class of BSDE, which guarantee the existence and the uniqueness of solutions.

1. The generator f should satisfy the following conditions:

a. The process $(f(t,0,0,0))_{t\leq T}$ belongs to $L^2(\Omega \times [0,T],\ dP \times dt)$.

b. There exists a constant $k \in (0,\infty)$ such that for any $y,\ y',\ z,\ z' \in \mathbb{R}$ and $u,\ u' \in L^2(E, \mathcal{E}, \lambda, \mathbb{R})$ we have,

$$|f(\omega,t,y,z,u) - f(\omega,t,y,z,u')| \leq k(|y-y'| + |z-z'| + \|u-u'\|\ \ P-a.s. \tag{4}$$

where $\|u\| = \int_E |u(e)|\lambda(de)$.

2. Definition of a closed subset $\mathcal{K} \subset \mathbb{R}^{1+d}$, viz

$$\mathcal{K} = \{(y,z) \in \mathbb{R}^{1+d} : \phi(t,y,z) \geq 0,\ \forall\ \ t \in [0,T]\}, \tag{5}$$

whose support function is given by

$$\delta(\alpha,\beta) = \sup_{(y,z)\in\mathcal{K}} (\alpha y + \beta z), \quad (\alpha,\beta) \in \mathbb{R}^{1+d}, \tag{6}$$

which is continuous on its effective domain

$$\tilde{\mathcal{K}} = \{x \in \mathbb{R}^{1+d} : \delta(x) < \infty\}. \tag{7}$$

2.2. Existence and uniqueness: the general case

In this section, based on a classical penalization argument, as it has been used in [7] for example, we prove existence of a minimal solution to (2)-(3). For each $n \in \mathbb{N}$, we invoke the following penalized equation

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + n\int_t^T \phi^-(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds,de), \tag{8}$$

where $\phi^-(s, Y_s^n, Z_s^n) = \max(-\phi(s, Y_s^n, Z_s^n), 0)$, is the negative part of function ϕ. Under standard conditions on the coefficients f and ϕ, we know from the theory of BSDEs that equation (8) admits a unique solution (see e.g. Hamadène and Ouknine [11]). In what follows, we prove the nondecreasing monotonicity of the sequence $(Y^n)_n$.

Theorem 2.1. (Comparison Theorem) *The sequence $(Y^n)_n$ is nondecreasing, i.e for all $n \in N$,* $Y_t^n \leq Y_t^{n+1},\ a.s.,\ t \in [0,T]$.

Proof. Recall our penalized BSDE with jumps (8) when $f^n(s,Y,Z,\phi) = f(s,Y,Z) + n\phi^-(s,Y,Z)$.

Since $f^n \leq f^{n+1}$, we can apply the comparison theorem 4.1 of [7] and obtain the required result. ■

Since, we have the same penalized BSDEs with jump, as in Essaky [9], then we can omit the following monotonic limit theorem when the noise is driven by a Brownian motion and an independent Poisson point process.

Theorem 2.2. (Monotonic limit theorem) *Assume that f satisfies the conditions noted above,* $\xi \in L^2(\Omega, F, P)$*, and* K^n *a continuous increasing process such that* $\sup_{n\in\mathbb{N}} \mathbb{E}(K_t^n)^2 < \infty$*, and* $K_0^n = 0$*, for any* $n \in N$ *and* $t \in [0,T]$*. Let* (Y^n, Z^n, V^n) *be the solution of the following BSDE*

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds, de), \tag{9}$$

where $\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T |Z_s^n|^2 ds < \infty$ *and* $\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \int_E (V_s^n(e))^2 \lambda(de)ds < \infty$*. If* $(Y^n)_n$ *converges increasingly to* Y *with* $\mathbb{E} \sup_{t\in[0,T]} |Y_t|^2 < \infty$*, then there exist* $Z \in L^2(B)$*,* $K \in A^2$ *and* $U \in L^2(\tilde{\mu})$ *such that, the triplet* (Z, K, U) *satisfies the following equation*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de),$$

where :

$$Z^n \rightharpoonup Z \text{ in } L^2(B),$$

$$K^n \rightharpoonup K \text{ in } L^2(\mathcal{F}_t),$$

$$V^n \rightharpoonup V \text{ in } L^2(\tilde{\mu}).$$

Moreover, for every $p \in [1,2]$, the following strong convergence holds.

$$\mathbb{E}\left[\int_0^T |Y_s^n - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^p ds + \int_0^T \left(\int_E |V_s^n - V_s|^2 \lambda(de)\right)^{\frac{p}{2}} ds\right] \to 0.$$

Inspired by Peng's definition [16], let us redefine the minimal solution of our portfolio-constrained BSDEs with jump, as the smallest super-solution of the same BSDEs type.

Definition 2.2. (Smallest super-solution) We say that Y_t is the smallest super-solution of (2)-(3) on $[0,T]$ with $Y_T = \xi$ and decomposition (Z_t, K_t, V_t), if for any $\hat{Y}_t$, we have $Y_t \leq \hat{Y}_t$ a.s, where $\hat{Y}_t$ is super-solution of (2)-(3) on $[0,T]$ with $\hat{Y}_T = \xi$, and decomposition $(\hat{Z}_t, \hat{K}_t, \hat{V}_t)$.

To proof existence of smallest super-solution of (2)-(3), we follow the same steps as in [16]. Then recall our penalized BSDE with jumps (8).
We define then for each $n \in \mathbb{N}$,

$$K_t^n = n\int_0^t \phi^-(s, Y_s^n, Z_s^n)ds\,; \quad 0 \leq t \leq T, \tag{10}$$

which is a nondecreasing process in $\mathbf{A}^2$.

The rest of this section is devoted to the convergence of sequence $(Y^n, Z^n, V^n, K^n)_n$ to the smallest super-solution we are interested in. To this aim we define $Y_t := \liminf_{n\to\infty} Y_t^n$, and we assume the hypothesis that follows.

H 2.1. There exist a quadruple $(\breve{Y}, \breve{Z}, \breve{V}, \breve{K}) \in \mathcal{S}^2 \times L^2(B) \times L^2(\tilde{\mu}) \times \mathbf{A}^2$ satisfying (2)-(3).

Theorem 2.2. *The limit (Y, Z, V, K) is the smallest super-solution of constrained BSDE with jump (2)-(3), for which*

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] < \infty.$$

Proof. Under assumption (1), we have $\breve{Y}$ as a super-solution of (2)-(3) with decomposition $(\breve{Z}, \breve{K}, \breve{V})$, it follows that

$$\|\breve{Y}\|_{\mathcal{S}^2} < \infty. \tag{11}$$

In fact, $\breve{Y}$ can be regarded as the solution of the following BSDE:

$$\breve{Y}_t = \xi + \int_t^T \breve{f}(s, \breve{Y}, \breve{Z}, \phi^-)ds + \breve{K}_T - \breve{K}_t - \int_t^T \breve{Z}_s dB_s - \int_t^T \int_E \breve{V}_s(e)\tilde{\mu}(ds, de), \tag{12}$$

where $\breve{f}(s, \breve{Y}, \breve{Z}, \phi) = f(s, \breve{Y}, \breve{Z}) + n\, \phi^-(s, \breve{Y}, \breve{Z})$.

According to the comparison theorem 2.1, since $n\, \phi^-(s, \breve{Y}, \breve{Z})$ is a nondecreasing process, we conclude that $Y_t^n \le \breve{Y}_t$. Also by the same theorem $Y_t^n \le Y_t^{n+1}$. Thus, Y_t^n converges increasingly to $Y_t \le \breve{Y}_t$.

It remains to show that

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] < \infty.$$

We notice however that Y_t is bounded from above by any super-solution satisfying constraint (3) and particularly $\breve{Y}$, which satisfies

$$\|\breve{Y}\|_{\mathcal{S}^2} < \infty.$$

Then Y_t is a super-solution and

$$\|Y\|_{\mathcal{S}^2} < \infty.$$

By the monotonic limit theorem 2.2, we have $Z^n \to Z$ in $L^p(B)$ and $V^n \to V$ in $L^p(\tilde{\mu})$ for $p \in [0, 2[$. Then

$$\phi^-(Y_n, Z_n) \to \phi^-(Y, Z) \text{ in } \mathbf{L}^p_{\mathbb{F}}.$$

From other side, and since Y_t is a super-solution, there exist a constant C such that

$$\mathbb{E}(K_T^n)^2 = n^2 \mathbb{E}\left[\int_0^T \phi^-(Y^n, Z^n)\right]^2 \le C, \tag{13}$$

meaning that $K \in \mathbf{A}^2$ and $\phi^-(Y^n, Z^n) \to 0$ in $\mathbf{L}^p_{\mathbb{F}}(0, T)$. This implies that the constraint (3) is satisfied. ■

Remark 2.1. (Existence of solution when f depends on V) In the case of the generator f

depending on (Y,Z,V), the main argument to prove the existence will be the fixed point theorem, by choosing an suitable mapping. However, since the constraint (3) is not easy to handle, we cannot apply directly this method on BSDEs of type (2)-(3). So the key idea here is to prove existence in a penalized BSDE case by the fixed point theorem. In view of that the process K_t^n could be chosen as:

$$K_t^n := n\int_0^t \rho(Z_s^n)ds, \tag{14}$$

where $\rho(z) := dist(z,K)$, this process will be very useful in proving contraction of our map. By some argument of convergence, may be we can prove that the solution of penalized BSDEs converges to a minimal solution of BSDE (2)-(3), without additional conditions on the generator f.

2.3. **A particular case**: **the Markovian setting**

Here we deal with the Markovian case of a BSDE with jump (2)-(3), for which we will prove existence and uniqueness. In such a case, f depends only on the the forward process X. Our portfolio-constrained BSDE with jump takes the form:

$$Y_t = \xi + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T\int_E V_s(e)\tilde{\mu}(ds,de), \tag{15}$$

$$\phi(t,Y_t,Z_t) \geq 0 \text{ a.s., a.e.} \tag{16}$$

Let $\mathcal{D}$ be the set of an $\mathbb{F}$-progressively measurable bounded process $\nu : [0,T]\times\Omega \to \tilde{\mathcal{K}}$, with $\mathbb{E}\int_0^T \|\nu\|^2 ds < \infty$. For $\nu := (\nu^1,\nu^d)$, consider the probability measure $\mathbf{P}^\nu$ equivalent to $\mathbf{P}$ on $(\Omega,\mathcal{F}_T)$, given by

$$\frac{d\mathbf{P}^\nu}{d\mathbf{P}} = \exp\left(\int_0^T \nu^d(s)' dB_s - \frac{1}{2}\int_0^T \|\nu^d\|^2 ds\right) \tag{17}$$

Then, by Girsanov transformation, the new stochastic process $B_t^\nu = B_t - \int_0^t \nu^d(s)ds$ is Brownian motion under probability $\mathbf{P}^\nu$. Notice that, since B and μ are independent, this change of probability measure does not change the Poisson measure $\mu^\nu = \mu$. The main result of this particular setting is reported next. Its proof is reported in the appendix.

Theorem 2.4. *There exist a unique minimal solution* $(Y,Z,V,K) \in S^2 \times L^2(B) \times L^2(\tilde{\mu}) \times A^2$, *with K predictable, and Y has the explicit functional representation*

$$Y_t = ess\sup_{\nu\in\mathcal{D}} \mathbb{E}^\nu\left[\xi + \int_t^T [f(X_s) - \delta(\nu(s))]ds \Big/ \mathcal{F}_t\ .\right] \tag{18}$$

3. Application to Finance

3.1. **Constrained-hedging strategies of American options**

In this section, we apply the result of previous sections to construct an option pricing problem, which is called *American option*. A similar contingent claim to European option, but can be exercised by the buyer anytime during its life. Let us now introduce a nonnegative process $S_. \in \mathcal{S}^2$, with continuous path such that $S_T \leq \xi$ a.s., and consider the new problem

$$Y_t = \xi + \int_t^T f(X_s)ds + K_T - K_t + K'_T - K'_t - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds,de), \tag{19}$$

$$\phi(Z_t) \geq 0 \quad a.s., \tag{20}$$

$$Y_t \geq S_t, \forall t \in [0,T] \; and \int_0^T (Y_t - S_t)dK'_t = 0. \tag{21}$$

The pair (ξ, S, ϕ) could be seen as an American contingent claim, with constraint ϕ on the hedging strategy. The solution $(Y,Z,V,K,K') \in \mathcal{S}^2 \times L^2(B) \times L^2(\mu) \times A^2 \times A^2$ of such a problem will provide the minimal hedging price of the contract, which is defined to be the infimum of the initial wealth amount Y_0, such that the seller can deliver the payoff, without having to use additional outside funds. Y is interpreted as the wealth of an investor who wants to hedge an American contingent claim, with some hedging strategy Z that verifies constraint ϕ, and K, K' are the cumulative consumptions to keep our constraints verified. Finally, V represents the jumps, issued from the forward stochastic differential equation of the price. Our, purpose is to determine the minimal hedging price. To begin, let us provide a definition of an admissible solution.

Definition 2.2. (Admissible solution) An admissible solution (Y,Z,V,K,K') of (19)-(21) is called minimal solution if for any other solution $(\tilde{Y},\tilde{Z},\tilde{V},\tilde{K},\tilde{K}')$ of (19)-(21), it holds that :

$$Y_t \leq \tilde{Y}_t, \quad t \in [0,T], \; P-a.s.$$

In order to solve (19)-(21), we need to introduce the penalized BSDE, for every $n \in \mathbb{N}$,

$$Y_t^n = \xi + \int_t^T f(X_s)ds + K_T^n - K_t^n + K_T^{'n} - K_t^{'n} - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds,de), \tag{22}$$

where

$$K_t^n = n\int_0^t \rho(Z_s^n)ds,$$

and $\rho(z)$ denotes the distance $z \in \mathbb{R}^d$ from the set $\mathcal{K}$, which satisfies the Lipschitz conditions. Also

$$K_t^{'n} = n\int_0^t (Y_s^n - S_s)^- \, ds,$$

where $y \to n(y - S_s)^-$ is Lipschitz w.r.t y. From the standard theory of BSDEs with jump, equation (22) should admit a unique solution (Y,Z,V,K,K') In order to derive a functional representation of such penalized BSDEs, we begin by proving the comparison theorem that follows.

Lemma 3.1. (Comparison theorem) *The sequence* $(Y^n)_n$ *is nondecreasing, i.e :* $Y^n \leq Y^{n+1}$ $a.s.$ $\forall n \in N, \; 0 \leq t \leq T.$

Proof. Recall the penalized BSDE (22) in the form

$$Y_t^n = \xi + \int_t^T F^n(s, X_s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds, de), \tag{23}$$

where $F^n(s,x,y,z) = f(x) + n(y - S_s)^- + n\rho(z)$.
Since $y \to (y-S)^-$ and $z \to n\rho(z)$ are Lipschitz, then F is Lipschitz. Furthermore $F^n \leq F^{n+1}$ since :

$$n(y-S) \leq (n+1)(y-S)$$

and

$$n\rho(z) \leq (n+1)\rho(z)$$

Apply then the comparison theorem 4.1 of [7] to arrive at the required result. ■

Theorem 3.1. *The penalized BSDE (22) admits a unique solution Y_t^n satisfying the explicit functional representation*

$$Y_t^n = ess \sup_{\nu \in \mathcal{D},\, \tau \in \Gamma_t} \mathbb{E}^\nu \left[\xi \mathbf{1}_{\tau = T} + (Y_\tau^n \wedge S_\tau)\mathbf{1}_{\tau < T} + \int_t^\tau (f(X_s) - \delta(\nu(s)))ds / \mathcal{F}_t \right], \tag{24}$$

where D is the set of F-progressively measurable bounded process $\nu : [0,T] \times \Omega \to \tilde{K}$, with $\mathbb{E}\int_0^T \|\nu\|^2 ds < \infty$, and Γ is the set of all stopping times dominated by T, viz

$$\Gamma_t = \{\tau \in \Gamma;\ t \leq \tau \leq T\}.$$

Proof. Actually, for any $n \geq 0$ and $t \geq 0$, we have:

$$Y_t^n = \xi + \int_t^T f(X_s)ds + n\int_t^T \rho(Z_s^n)ds + n\int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s$$
$$- \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds, de)$$

$\Leftrightarrow$

$$Y_t^n + \int_t^T \delta(\nu(s))ds = \xi + \int_t^T f(X_s)ds + \int_t^T [n\rho(Z_s^n) - Z_s^n \nu(s) + \delta(\nu(s))]ds$$
$$+ n\int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s^\nu - \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds, de).$$

From another side, it follows from lemma 3.1, of [5], that

$$\sup_{z \in \mathbb{R}^d} [\nu' z - n\rho(z)] = \begin{cases} \delta(\nu), & \nu \in \tilde{\mathcal{K}} \cap B_n, \\ \infty, & \nu \notin \tilde{\mathcal{K}} \cap B_n, \end{cases} \tag{25}$$

where $B_n := \{\nu \in \mathbb{R}^d; \|\nu\| \leq n\}$. Thus

$$Y_t^n + \int_t^T \delta(\nu(s))ds \geq \xi + \int_t^T f(X_s)ds - n\int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s^\nu$$
$$- \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds, de).$$

Therefore, for any stopping time $\tau \geq t$, and by taking the conditional expectation in both sides we can write

$$Y_t^n \geq \mathbb{E}^\nu\Big[Y_n^\tau + \int_t^\tau [f(X_s) - \delta(\nu(s))]ds + n\int_t^\tau (Y_s^n - S_s)^- ds/\mathcal{F}_t \Big]$$
$$\geq \mathbb{E}^\nu\Big[\xi \mathbf{1}_{\tau=T} + (Y_n^\tau \wedge S_\tau)\mathbf{1}_{\tau<T} + \int_t^\tau [f(X_s) - \delta(\nu(s))]ds/\mathcal{F}_t \Big]. \quad (26)$$

In order to prove this inequality, let us define τ^* as a stopping time such that

$$\tau^* = \inf\{s \geq t \,/\, K_s^{'n} - K_t^{'n} > 0\} \wedge T, \quad (27)$$

where $K_t^{'n} = n\int_0^t (Y_s^n - S_s)^- ds$. Notice that $K_{\tau^*}^{'n}(w) = K_t^{'n}(w)$ follows from the continuity of K^n, and according to Hamadène and Ouknine [12], we have $Y_{\tau^*}^n \mathbf{1}_{\tau^*<T} = (Y_{\tau^*}^n \wedge S_{\tau^*})\mathbf{1}_{\tau^*<T}$. Moreover, by Dellacherie Meyer [6], there exists a $\nu^* \in \tilde{\mathcal{K}}$ such that

$$n\rho(z) - z\nu^* + \delta(\nu^*) = 0, \ a.e.$$

All of this allows for

$$Y_t^n + \int_t^{\tau^*} \delta(\nu^*(s))ds = Y_{\tau^*}^n + \int_t^{\tau^*} f(X_s)ds - \int_t^{\tau^*} Z_s^n dB_s^{\nu^*} - \int_t^{\tau^*}\int_E V_s^n(e)\tilde{\mu}(ds,de)$$
$$\Leftrightarrow$$
$$Y_t^n = (Y_{\tau^*}^n \wedge S_{\tau^*})\mathbf{1}_{\tau^*<T} + \xi\mathbf{1}_{\tau^*=T} + \int_t^{\tau^*} [f(X_s) - \delta(\nu^*(s))]ds - \int_t^{\tau^*} Z_s^n dB_s^{\nu^*}$$
$$-\int_t^{\tau^*}\int_E V_s^n(e)\tilde{\mu}(ds,de).$$

Taking the conditional expectation and using inequality (26) bring this proof to to its end. ■

We conclude from above proposition and comparison theorem (lemma 3.1), that the limit of Y_t^n exists almost surely and we denote it as

$$Y_t = \lim_{n\to\infty} Y_t^n \, . \quad (28)$$

Following is a prove that this limit leads to minimal solution of our problem, which provides for a functional representation for it.

Corollary 3.1. *The process Y_t is the unique minimal solution of problem (19)-(21), satisfying the functional representation*

$$Y_t = ess\sup\nolimits_{\nu\in\mathcal{D},\, \tau\in\Gamma_t} \mathbb{E}^\nu\Big[\xi\mathbf{1}_{\tau=T} + S_\tau\mathbf{1}_{\tau<T} + \int_t^\tau (f(X_s) - \delta(\nu(s)))ds/\mathcal{F}_t \Big],$$
$$Y_T = \xi, \ and \ Y_t \in S^2.$$

Proof. Let the process $\tilde{Y}$, defined by

$$\tilde{Y}_t = ess \sup_{\nu\in\mathcal{D},\, \tau\in\Gamma_t} \mathbb{E}^\nu\Big[\xi\mathbf{1}_{\tau=T} + S_\tau\mathbf{1}_{\tau<T} + \int_t^\tau (f(X_s) - \delta(\nu(s)))ds/\mathcal{F}_t \Big], \quad (29)$$

be a solution of (19)-(21). In fact, since $S \in \mathcal{S}^2$, $f \in \mathbb{R}$, δ is continuous on its effective domain and ξ is square integrable, it follows that $\tilde{Y}_t \in \mathcal{S}^2$. From the functional representation of Y^n, we have for all $n \in \mathbb{N}$

$$Y^n_t \le \tilde{Y}_t, \;\; 0 \le t \le T, a.s. \tag{30}$$

Also by comparison theorem (lemma 3.1), it follows that $Y^n_t \nearrow Y_t \le \tilde{Y}_t$. From other side $\left(Y^n_t + \int_0^t (f(X_s) - \delta(\nu(s)))\right)_{t\le T}$ is a rcll super-martingale and converges increasingly to $\left(Y_t + \int_0^t (f(X_s) - \delta(\nu(s)))\right)_{t\le T}$, which is also rcll super-martingale (See Dellacherie Meyer [6]). Thus $Y_t \in \mathcal{S}^2$ since it is dominated by $\tilde{Y}_t$. It remains to derive the functional representation of Y_t in order to complete this proof.

By slight modification of the developments by Hamadène and Ouknine [12], we obtain from (25):

$$\mathbb{E}[Y^n_0] = \mathbb{E}\left[\xi + \int_0^T f(X_s)ds\right] + \mathbb{E}\left[\int_0^T n\rho(Z^n_s)ds + n\int_0^T (Y^n_s - S_s)^- ds\right]. \tag{31}$$

Furthermore, by the monotonic limit theorem 2.2, of Essaky [9] when applied to penalized BSDE (25), the nondecreasing process (Y^n_t) converges increasingly to:

$$Y_t = \xi + \int_t^T f(X_s)ds + K_T - K_t + K'_T - K'_t - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de).$$

Moreover

$$\mathbb{E}\left[\int_0^T |Y^n_s - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z^n_s - Z_s|^p ds + \int_0^T \left(\int_E |V^n_s - V_s|^2 \lambda(de)\right)^{\frac{p}{2}} ds\right] \to 0.$$

Since dividing the right-hand side of (31) by n, and taking the limit as $n \to \infty$ lead to

$$\mathbb{E}\int_0^T \rho(Z_s)ds = 0, \quad \mathbb{E}\int_0^T (Y_s - S_s)^- ds = 0,$$

it follows that for any $n \ge 0$ and $t \le T$, we have $\rho(Z_t) = 0$; meaning that $Z_t \in \mathcal{K}$ and $\phi(Z_t) \ge 0$ $a.s.$ Also, from $\mathbb{E}\int_0^T (Y_s - S_s)^- ds = 0$, we have, since Y and S are rcll, that $Y_t \ge S_t$ for any $t \le T$, $(Y_T = \xi)$.

As a direct consequence of the continuity of the Snell envelope (See appendix [A1] in [12]), the sequence $\left(Y^n_t + \int_0^t (f(X_s) - \delta(\nu(s)))\right)_{t\le T}$ equals to

$$ess \sup_{\nu\in\mathcal{D},\, \tau\in\Gamma_t} \mathbb{E}^\nu\left[\xi\mathbf{1}_{\tau=T} + (Y^n_\tau \wedge S_\tau)\mathbf{1}_{\tau<T} + \int_0^\tau (f(X_s) - \delta(\nu(s)))ds/\mathcal{F}_t\right],$$

and converges to

$$ess \sup_{\nu\in\mathcal{D},\, \tau\in\Gamma_t} \mathbb{E}^\nu\left[\xi\mathbf{1}_{\tau=T} + S_\tau\mathbf{1}_{\tau<T} + \int_0^\tau (f(X_s) - \delta(\nu(s)))ds/\mathcal{F}_t\right],$$

which is equal to $\left(Y_t + \int_0^t (f(X_s) - \delta(\nu(s)))\right)_{t\le T}$. ■

3.2. **Constrained-hedging strategies of American game options**

In this part, we apply the theory developed in subsection 3.1, to another problem in finance: hedging *American game options* with constraint on the hedging strategy. To begin let us consider the following constraint-BSDE with two-reflections

$$\begin{cases} Y_t = \xi + \int_t^T f(X_s)ds + K_T - K_t + \zeta_T - \zeta_t - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds,de), & (32) \\ \phi(Z_t) \geq 0 \ a.s., & (33) \\ L_t \leq Y_t \leq U_t, \ \forall t \in [0,T], \ and \int_0^T (Y_t - U_t)d\zeta_t^- = \int_0^T (Y_t - L_t)d\zeta_t^+ = 0, & (34) \\ K_0 = \zeta_0^+ = \zeta_0^- = 0 \ ; \ K, \ \zeta^+, \ \zeta^- \ are\ continuous\ nondecreasing. \end{cases}$$

Here, the process Y has to remain between L and U thanks to conditions on ζ. An American game option is unlike American option where only the buyer has the right to choose the exercise time, but also the seller has the right to force the exercise time. However, in order to avoid immediate exercise, it is required that the payoff be higher if the seller wants to exercise. Otherwise, if the seller does anything, the game option become a standard American option. Mathematically, let τ and ς be two stopping times, and redefine Γ_ς to be the set of stopping times τ such that $\tau \geq \varsigma$, and let $\tau \in \Gamma_\varsigma$ be the time the buyer chooses to exercise the option, and $\theta \in \Gamma_\varsigma$ be that of the seller. If $\tau \leq \theta$ then the seller pays $L(\tau, X_\tau)$, while if $\theta < \tau$ then the seller pays $U(\theta, X_\theta)$. In both cases the seller pays. However, if neither exercises the option by the maturity date T, then the seller pays $\xi = g(X_T)$.

In the unconstrained case, the value game of the buyer who wants to maximize the payoff $g(X_T)$ is done by $ess \inf_{\theta\in\Gamma_t} ess \sup_{\tau\in\Gamma_t} \vartheta_t(\theta,\tau)$, and the value game of a seller who wants to minimize the payoff is given by $ess \sup_{\tau\in\Gamma_t} ess \inf_{\theta\in\Gamma_t} \vartheta_t(\theta,\tau)$, where

$$\vartheta_t(\theta,\tau) := \mathbb{E}^\nu\Big[\xi \mathbf{1}_{\tau\wedge\theta=T} + L_\tau \mathbf{1}_{[\tau\leq\theta<T]} + U_\theta \mathbf{1}_{\theta<T} + \int_t^{\tau\wedge\theta} f(X_s)ds/\mathcal{F}_t\Big].$$

An American game option is typically a Dynkin game, where its value function on $[\varsigma, T]$, is an $(\mathcal{F}_t)_{t\leq T}$ - adapted process $(Y_t)_{\varsigma\leq t\leq T}$ such that $\forall t \in [\varsigma, T]$

$$Y_t = ess \inf_{\theta\in\Gamma_t} ess \sup_{\tau\in\Gamma_t} \vartheta_t(\theta,\tau) = ess \sup_{\tau\in\Gamma_t} ess \inf_{\theta\in\Gamma_t} \vartheta_t(\theta,\tau),$$

with ς a Γ_t-stopping time. Y_τ is called the value of the game option on $[\tau, T]$. In order to derive an explicit solution of the problem (32)-(34), we have to take into our account the upper barrier U, which we can derive by following the same steps as in the previous section. Then we may obtain the minimal solution viz

$$Y_t = ess\sup_{\nu\in\mathcal{D}} ess\inf_{\theta\in\Gamma_t} ess\sup_{\tau\in\Gamma_t} \vartheta_t(\nu,\theta,\tau), \tag{35}$$

where, the new definition of ϑ has the form

$$\vartheta_t(\nu,\theta,\tau) := \mathbb{E}^\nu\Big[\xi \mathbf{1}_{\tau\wedge\theta=T} + L_\tau \mathbf{1}_{[\tau\leq\theta<T]} + U_\theta \mathbf{1}_{\theta<T} + \int_t^{\tau\wedge\theta} (f(X_s) - \delta(\nu(s)))ds/\mathcal{F}_t\Big].$$

This triple optimization problem of mixed stochastic control/stopping type, ensures the minimal wealth to hedge the American game option. The maximization on ν guaranties the constraint (33) on the portfolio, and the minimization/maximization over the stopping-times τ/θ ensures that constraints (34) are justified.

As in section 3.1, and by analogy with theorem 3.1, we construct, through penalization BSDE of the doubly reflected constrained-BSDE, the following explicit formula for Y_t^n . This takes a more complicated form, for some suitable quadruple $(Y^n, Z^n, V^n, K^n, \zeta^n) \in \mathcal{S}^2 \times L^2(B) \times L^2(\mu) \times A^2 \times A^2$.

$$\begin{cases} Y_t^n = \xi + \int_t^T [f(X_s) + n\rho(Z_s^n)]ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(ds,de) + \zeta_T - \zeta_t, & (36) \\ L_t \leq Y_t^n \leq U_t,\ \forall t \in [0,T],\ and \int_0^T (Y_t^n - U_t)d\zeta_t^- = \int_0^T (Y_t^n - L_t)d\zeta_t^+ = 0, & (37) \\ \zeta^{n+},\ \zeta^{n-}\ are\ continuous\ nondecreasing. \end{cases}$$

By using the comparison theorem (lemma 3.1), we have $Y^n \leq Y^{n+1}\ a.s., \forall n \in \mathbb{N}$, $0 \leq t \leq T$, the process Y^n defined above satisfies the explicit formula

$$Y_t^n = ess\sup_{\nu \in \mathcal{D}}\ ess\inf_{\theta \in \Gamma_t}\ ess\sup_{\tau \in \Gamma_t} \vartheta_t^n(\nu, \theta, \tau), \tag{38}$$

where, the new definition of ϑ has the form

$$\begin{aligned} \vartheta_t^n(\nu,\theta,\tau) := \mathbb{E}^{\nu}[\xi \mathbf{1}_{\tau\wedge\theta=T} + (L_\tau \wedge Y_\tau^n)\mathbf{1}_{[\tau\leq\theta<T]} + U_\theta \mathbf{1}_{\theta<\tau} \\ + \int_t^{\tau\wedge\theta}(f(X_s) - \delta(\nu(s)))ds/\mathcal{F}_t \Big]. \end{aligned}$$

This supremum is attained by the triplet $(\nu^*, \theta^*, \tau^*)$, where a.e.

$$n\rho(z) - z\nu^* + \delta(\nu^*) = 0,$$

$$\theta^* = \inf\{s \geq t \,/\, Y_s^n = U_s\} \wedge T,$$

$$\tau^* = \inf\{s \geq t \,/\, \zeta_s^{n^+} - \zeta_t^{n^+} > 0\} \wedge T,$$

with which we have

$$Y_t^n = \mathbb{E}^{\nu^*}\Big[\xi \mathbf{1}_{\tau^*\wedge\theta^*=T} + (L_{\tau^*} \wedge Y_{\tau^*}^n)\mathbf{1}_{[\tau^*\leq\theta^*<T]} + U_{\theta^*}\mathbf{1}_{\theta^*<\tau^*} + \int_t^{\tau^*\wedge\theta^*} f(X_s) - \delta(\nu^*(s))ds/\mathcal{F}_t\Big].$$

These facts can be employed, by the same techniques of convergence of the Snell envelope, to show that $\lim Y_t^n \nearrow Y_t$.

Remark 3.1. The existence and uniqueness of solution have been proved for an unconstrained case by Hamadène and Hassani [10], when the barrier L and U is either regular or satisfy Mokobodski's condition.

4. Appendix

4.1. Proof of theorem 2.4

For any solution $(\hat{Y}, \hat{Z}, \hat{V}, \hat{K}) \in \mathcal{S}^2 \times L^2(B) \times L^2(\hat{\mu}) \times \mathbf{A}^2$ satisfying (15)-(16) we have

$$\begin{aligned} \hat{Y}_t + \int_0^t [f(X_s) - \delta(\nu(s))]ds + \hat{K}_t + \int_0^t \delta(\nu(s))ds \\ = \hat{Y}_0 + \int_0^t \hat{Z}_s dB_s + \int_0^t \int_E \hat{V}_s(e)\tilde{\mu}(ds,de), \end{aligned}$$

which can be written in terms of B^νas

$$\hat{Y}_t + \int_0^t [f(X_s) - \delta(\nu(s))]ds + \hat{K}_t + \int_0^t [\delta(\nu(s)) - \nu^d(s)'\hat{Z}_s]ds$$
$$= \hat{Y}_0 + \int_0^t \hat{Z}_s dB_s^\nu + \int_0^t \int_E \hat{V}_s(e)\tilde{\mu}(ds,de) \quad := Q_t.$$

Since ν is bounded, the density $\frac{d\mathbf{P}^\nu}{d\mathbf{P}}$ is in $L^2(P)$, then by employing the assumptions: $Z \in L^2(B)$, and $V \in L^2(\tilde{\mu})$, with the Cauchy-Schwartz inequality, we have

$$\mathbb{E}^\nu\left(\int_0^T \|\hat{Z}_s\|^2 ds\right)^{\frac{1}{2}} \leq \left(\mathbb{E}\left(\left\{\frac{d\mathbf{P}^\nu}{d\mathbf{P}}\right\}^2\right) \cdot \mathbb{E}\left(\int_0^T \|\hat{Z}_s\|^2 ds\right)\right)^{\frac{1}{2}} < \infty \qquad (A.1)$$

$$\mathbb{E}^\nu\left[\int_0^T \int_E |\hat{V}_t(e)|^2 \lambda(de)dt\right]^{\frac{1}{2}} \leq \left(\mathbb{E}\left(\frac{d\mathbf{P}^\nu}{d\mathbf{P}}\right)^2 \cdot \mathbb{E}\left(\int_0^T \int_E |\hat{V}_t(e)|^2 \lambda(de)dt\right)\right)^{\frac{1}{2}} < \infty \qquad (A.2)$$

Therefore Q_t has the P^ν-martingale property.
Moreover, $\hat{K}_. + \int_0^. [\delta(\nu(s)) - \nu^d(s)'Z_s]ds$ is an increasing process. Hence

$$\hat{Y}_t + \int_0^t [f(X_s) - \delta(\nu(s))]ds \qquad (A.3)$$

is P^ν-super-martingale, and hence

$$\hat{Y}_t \geq \mathbb{E}^\nu\left[\hat{Y}_T + \int_t^T [f(X_s) - \delta(\nu(s))]ds/\mathcal{F}_t\right],\ t \in [0,T]\ a.s. \qquad (A.4)$$

Then,

$$\hat{Y}_t \geq Y_t := ess\sup_{\nu\in\mathcal{D}} \mathbb{E}^\nu\left[\xi + \int_t^T [f(X_s) - \delta(\nu(s))]ds/\mathcal{F}_t\right],\ t \in [0,T]\ a.s. \qquad (A.5)$$

In order to find the minimal solution it is sufficient to show that $(Y,Z,V,K) \in \mathcal{S}^2 \times L^2(B) \times L^2(\hat{\mu}) \times \mathbf{A}^2$.
In fact, for $\nu \equiv 0$, we observe that $\mathbf{P}^\nu = \mathbf{P}$. This implies that $\bar{Y} \leq Y \leq \hat{Y}$, where

$$\bar{Y} = \mathbb{E}\left[\xi + \int_t^T f(X_s)ds/\mathcal{F}_t\right],\ t \in [0,T]\ a.s.$$

Thus, since $\bar{Y} \in \mathcal{S}^2$ and $\hat{Y} \in \mathcal{S}^2$, we deduce that $Y \in \mathcal{S}^2$.
Next we prove that the minimal solution admits a unique decomposition, and we proceed with a similar programming argument as in [8]. In fact, we have proven that the process

$$\tilde{Q}_t = Y_t + \int_0^t [f(X_s) - \delta(\nu(s))]ds$$

is a rcll $\mathbf{P}^\nu$-supermartingale, for all $\nu \in \mathcal{D}$. Then by the Doob-Meyer decomposition of supermartingales we have

$$\tilde{Q}_t = Y_0 + M_t^\nu - K_t^\nu, \qquad (A.6)$$

where $M_0^\nu = 0$, M^ν is a $\mathbf{P}^\nu$-martingale, and K^ν is a rcll $\mathbf{P}^\nu$-nondecreasing predictable process with $K_0^\nu = 0$. By the martingale representation theorem, in a Poissonian-Brownian space, for each M^ν, $\nu \in \mathcal{D}$, there exist two predictable processes Z^ν and V^ν such that

$$Q_t = Y_0 + \int_0^t Z_s^\nu dB_s^\nu + \int_0^t \int_E V_s^\nu(e)\tilde{\mu}(ds,de) - K_t^\nu. \qquad (A.7)$$

By comparing the decomposition of (A.7), under P^ν and P^0, and identifying the martingale parts and predictable finite variation parts, we obtain that: $Z^\nu = Z^0 := Z$, $U^\nu = U^0 := U$ and

$$K_t^v = K_t^0 - \int_0^t Z_s v^d(s) ds. \qquad (A.8)$$

We are able then to rewrite (A.7) as

$$Y_t = Y_T + \int_t^T f(X_s) - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de) + \tilde{K}_T - \tilde{K}_t, \qquad (A.9)$$

where

$$\tilde{K}_t = K_t^v + \int_0^t \delta(v(s)) = K_t^0 + \int_0^t [\delta(v(s)) - Z_s v^d(s)] ds.$$

Moreover, the process $\tilde{Q}_t \in \mathcal{S}^2$, and by the decomposition (A.6), $M^v \in \mathcal{S}^2$ and $K^v \in \mathbf{A}^2$, implies that $Z \in L^2(B)$, $V \in L^2(\tilde{\mu})$ and $\tilde{K}^0 \in \mathbf{A}^2$. Finally since $K^v \in \mathbf{A}^2$ we deduce that $\tilde{K} \in \mathbf{A}^2$, which is still a nondecreasing process. ■

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