Journal of Numerical Mathematics and Stochastics, **13** (1) : 57-67, 2022
http://www.jnmas.org/jnmas13-3.pdf
Euclidean Press, LLC
Online: ISSN 2151-2302

Note on an Open Conjecture in Rational Dynamical Systems

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Abstract. *Recently, an increased interest has been witnessed in studying the theory of discrete dynamical systems, specifically of their associated difference equations. Sizable number of works on the behavior and properties of pertaining solutions (boundedness and unboundedness) have been published in various areas of applied mathematics and physics. One of the most important discrete dynamical models, which has attracted attention of researchers in the field, is the rational dynamical system. In this paper we give a negative answer to the eighth open conjecture in rational dynamical systems, proposed many years ago, by G. Ladas and Palladino, which states : Assume* $\alpha, \beta, \lambda \in [0, \infty)$. *Then every positive solution of the difference equation*:

$$z_{n+1} = \frac{\alpha + z_n \beta + z_{n-1} \lambda}{z_{n-2}}, \quad n = 0, 1, \ldots$$

is bounded if and only if $\beta = \lambda$. *Our negative answer is of general nature and is based on a construction of a subenergy function and some properties of Todd's difference equation. Some new results and arguments regarding that conjecture (Chebyshev approximation) are also presented.*

Key words: Difference Equation, Super-Energy Function, Rational Dynamical System, Boundedness.

AMS Subject Classifications: 39A10

1. Introduction

The theory of difference equations finds many applications in almost all areas of natural sciences [1]. That reflects the fundamental role that difference equations, with discrete and/or continuous arguments, play in modeling nonlinear dynamics and phenomena. They are also used in the area of combinatorics and in the approximation of solutions to partial differential equations [2]. The increased interest in difference equations is partly due to their ease of handling. Minimal computing times and graphical tools can be employed to exhibit how the solution of difference equations trace their bifurcations with changing their parameters [3]. This provides for a comprehensive understanding of their solution behavior, and reveals their pertaining invariant manifolds, particularly for nonlinear dynamical systems.

Let us define the sequence: $x = x_n$, $n \in \mathbb{Z}^+$ when every term is related to the following recurrence relation

$$x_n = f(n, x_{n-1}, x_{n-2}, x_{n-3}, \cdots x_{n-k}), \tag{1}$$

with a fixed $k > 0$.The autonomous variable n changes discontinuously, and the formula defined by (1) is called a difference equation with discrete arguments. If x is a function of continuous argument $m \in \mathbb{R}^+$ then the relation:

$$x(m) = f(t, x(m-1), x(m-2), \cdots x(m-k)), \tag{2}$$

is a difference equation with a continuous argument. In practice though, time, n, usually plays the role of the autonomous variable, which allows us to speak, respectively, of difference equations with discontinuous and continuous time. Discrete time equations arise when the quantity x under consideration is recorded at some interval of time. For example, if x is the relative abundance (compactness, density) of any biological nature, then for such an interval, it is sometimes recommended to consider the life time of one generation. On other occasions, the relationship between x_n and x_{n-1} is given satisfactorily by the first-order difference equation

$$x_n = \lambda x_{n-1}(1 - x_{n-1}), \tag{3}$$

(values x_n, as population density, should not fall out of the interval $[0,1]$. Therefore, as can be easily seen, the parameter x, being the coefficient of reproduction, should be enclosed in $[0,4]$), [4]. If $0 < \lambda < 1$ then the population goes to zero at a rate of a geometric sequence. If $1 < \lambda < 4$, then the comport of x_n can be duplicated (perhaps stabilize over time or become cyclic), or be very complex (chaotic). Complexities in the behavior of x_n arise due to the nonlinearity on the right side of equation (3). In this regard, :the following segment of increase (*interval* $(0, \frac{1}{2})$) is followed by the segment (*interval* $(\frac{1}{2}, 1)$)of descent. It turns out that by studying a real physical problem, it is more convenient to first derive relations for finite differences, then to make a passage to the limit to end up with differential equations. Only then, by discretization in time and space, one can arrive at the correct difference schemes, [5]. Due, in part, precisely to these reasons, the development of the theory of difference equations, starting from the end of the XVIII century, has gradually been lagging behind the rapidly and multifaceted developing theory of differential equations in ordinary, as well as in partial derivatives.

On anather note, attempts to understand the mechanisms of turbulence from the infinitesimal point of view have been encountering various obstacles [6], caused by the need to solve the Navier-Stokes equations or other nonlinear equations, not inferior to them in complexity. To clarify the properties of turbulence, completely different equations, has been investigated, to reveal its discrete nature, which has lately become increasingly obvious. By these, we mean such features of a disturbance, such as intermittence, the construction of various kinds of consistent structures, such as cyclones, etc. Recently identified directions in structural turbulence, [7], confirms the adequacy of the suggested opinion on the essence of the previous phenomenon. Indeed, from the properties of their solutions, one can surprisingly clearly guess many features of turbulence. Expectably, modeling of turbulence by difference equations is simpler than modeling by differential equations. In this regard, we note that in the book, [7], on modeling sequences of operations for the formation of cyclones and vortices of decreasing size. These are developed within the theory of hydrodynamic systems. Each of the considered systems may include nonlinear ordinary differential equation, to obtain the subsequent scale of the vortices, where the order increases by three folds. As a result, the dimension of such a system may grow catastrophically to become impossible to handle. Simultaneously, the mechanism of the cascade operation itself can be implemented already within the framework of only one dynamic equation of the form (3). A large number of interesting works (published papers, books, notes, ect.) happen to exist, [8], in the field of discrete dynamical systems, to determine the behavior (boundedness) of the solutions of a rational difference equations of the following form:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B\, x_n + C x_{n-1} + D x_{n-1}} \quad , \tag{4}$$

with nonnegative parameters $\alpha, \beta, \gamma, \delta, A, B, C$ and D. The main objective of studying the boundedness properties of the solutions of the dynamics defined by (4) is to check and to prove whether solutions are still bounded for all positive initial conditions, or if there exist some positive initial conditions where the solutions are unbounded. It should be noted that the book Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures, [9] & [10], deals with a large class of difference equations described by Equation (4). Some open problems related to (4), in which the boundedness properties were not known, have recently been solved in [11], by the following assumption $\delta = A = B = C = 0$, with the variable change $x_n \to \frac{x_n}{D}$, with $\alpha \geq 0, \beta > 0, \gamma > 0, D > 0,$ when equation (4) reduces to the following form :

$$x_{n+1} = \frac{D\alpha + x_n\beta + x_{n-1}\gamma}{x_{n-2}}, \quad n = 0, 1, \ldots, D\alpha = \alpha'. \tag{5}$$

It is shown in a paper by Lugo and Palladino, [12], that there exist unbounded solutions to (5) in the case of $0 \leq \alpha < 1$ and $0 < \beta < \frac{1}{3}$. Ying Sue Huang and Peter M. Knopf showed in [11] for $\alpha' \geq 0, \beta > 0$ and if $\beta \neq 1$ there exist positive initial conditions such that the solutions are unbounded except for the case $\alpha' = 0$ and $\beta > 1$. In this paper we shall disprove the "only if" part of the eight conjecture of G. Ladas, Lugo and Palladino [12]. In particular, using a subenergy function, we shall demonstrate computationally (using a Mathematica Code) up to 10^{40} solutions that we have :

$$z_{n+1} = \frac{\alpha + z_n\, \beta + z_{n-1}\, \lambda}{z_{n-2}}, \quad n = 0, 1, \ldots, \tag{6}$$

implying $\beta = \gamma$ is true, but the converse is not.

2. Main Result

Conjecture 2.1. *Assume* $\alpha, \beta, \lambda \in [0, \infty)$. *Then every positive solution of the difference equation (6) is bounded if and only if* $\beta = \lambda$.

Proof. Suppose that $\beta = \lambda > 0$. Let $x_n := z_n/\beta$ and $c := \alpha/\beta^2$. Then the dynamics for (6) can be rewritten viz

$$x_{n+1} = \frac{c + x_n + x_{n-1}}{x_{n-2}}, \tag{7}$$

(say for $n = 2, 3, \ldots$), just with one parameter $c \geq 0$. the dynamial equation (7) is exactly Todd's difference equation (or simply Todd's equation) which possesses the invariant :

$$(c + x_n + x_{n-1} + x_{n-2})\left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_{n-2}}\right) = \text{constant} . \tag{8}$$

The invariants of difference equations play an important role in understanding the stability and qualitative behavior of their solutions. To be more precise, if the invariant is a bounded manifold [13], then the solution is also bounded. Recently Hirota et al. [14] found two conserved quantities H_n^1 and H_n^2 for the third- order Lyness equation. It should be noted that Lyness equation is a special case of equation (7) such that $c = 1$. The two quantities are independents, and one of the conserved quantities is the same form as that of (8). Both of two conserved quantities formula were derived from discretization of an anharmonic oscillator namely using its equation of its motion see the first equation here, [14], we may consider those conserved quantities as a conserved subenergy of anharmonic oscillator, this means that (8) presents a sub energy function of that anharmonic oscillator. To prove the "if" part of the conjecture it would be enough to construct for each nonnegative c, a "subenergy" function [15] $f_c : (0, \infty)^3 \to \mathbb{R}$ such that :

$$f_c(x_0, x_1, x_2) \to \infty \quad \text{as} \quad x_0 + x_1 + x_2 \to \infty \tag{9}$$

Note that the subenergy function is the invariant of the third difference equation, namely equation (7), if we assume, when $n \geq 0$, that :

$$f_c(x_n, x_{n-1}, x_{n-2}) = (c + x_n + x_{n-1} + x_{n-2})\left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_{n-2}}\right)$$
$$= \text{constant}, \tag{10}$$

then the condition (9) is satisfied in (10), see Lemma 2 in ([16], p.4). For the RHS of (10) see also Theorem 2.1 in ([16], p.31). Moreover, since the invariant of the dynamical equation (7) is constant, then f_c could be referred to as the conservation of energy along the path of the dynamical system. For some natural k and all $x = (x_0, x_1, x_2) \in (0, \infty)^3$ one has the "subenergy" inequality $f_c(T^k x) \leq f_c(x)$, where $Tx := (x_1, x_2, x_3)$, with $x_3 = \frac{c + x_2 + x_1}{x_0}$, in these dynamics. Of course, T^k is the kth power of the operator T. For $k = 1$, the sub-energy inequality is the functional inequality

$$f_c\left(x_1, x_2, \frac{c + x_2 + x_1}{x_0}\right) \leq f_c(x_0, x_1, x_2) \quad \text{for all positive } x_0, x_1, x_2. \tag{11}$$

To construct a subenergy function, one might want to start with some easy function $f_{c,0}$ such that $f_{c,0}(x_0, x_1, x_2) \to \infty$ as $x_0 + x_1 + x_2 \to \infty$, and then consider something like $f_{c,0} \vee (f_{c,0} \circ T^k) \vee (f_{c,0} \circ T^{2k}) \vee \ldots$. Inequality (11) can be clearly restated in the following more symmetric form:

$$x_0\, x_3 = c + x_1 + x_2 \Rightarrow f_c(x_1, x_2, x_3) \leq f_c(x_0, x_1, x_2), \tag{12}$$

for all positive real x_0, x_1, x_2, x_3. condition $x_0 + x_1 + x_2 \to \infty$ in (7) can be replaced by any one of the following (stronger) conditions: (i) $x_0 \to \infty$ or (ii) $x_1 \to \infty$ or (iii) $x_2 \to \infty$; this of course will replace condition (7) by a weaker condition, which will make it easier to construct a sub-energy function f_c. Here are the details. Suppose that (11) holds for some function f_c such that $f_c(x_0,x_1,x_2) \to \infty$ as $x_0 \to \infty$. Additionally, assume nonetheless that a positive sequence $(x_0,x_1,\ldots)$ satisfying condition (5) is unbounded, so that as $k \to \infty$, one has $x_{n_k} \to \infty$ for some sequence (n_k) of natural numbers. Then $f_c(x_{n_k},x_{1+n_k},x_{2+n_k}) \to \infty$ as $k \to \infty$. This contradicts (4), which implies, by induction, that $f_c(x_n,x_{1+n},x_{2+n}) \le f_c(x_0,x_1,x_2)$ for all natural n. Quite similarly, one can deal with (ii) $x_1 \to \infty$ or (iii) $x_2 \to \infty$ in place of (i) $x_0 \to \infty$.

Moreover, instead of the dynamics of the triples (x_n,x_{1+n},x_{2+n}) one can consider the corresponding dynamics (in n) of the consecutive m-tuples $(x_n,\ldots,x_{m-1+n})$ for any fixed natural m.

Also, instead of inequality $f_c(x_1,x_2,x_3) \le f_c(x_0,x_1,x_2)$ in (7), one may consider a weaker inequality like $f_c(x_2,x_3,x_4) \le f_c(x_0,x_1,x_2) \vee f_c(x_1,x_2,x_3)$ for all positive $x_0,\ldots,x_4$ satisfying condition (5), thanks to the invariance of Todd's difference equation (2) which is defined in our case to be a subenergy function, such that it is easy to see that if part of the conjecture follows since the subenergy f_c is always accessible. In ([17], p.35) the authors showed that every positive solution of the dynamics (5) with an invariant are bounded and persistent. This result is the affirmation that the invariant must be a constant subenergy function which it is always found for all positive initial conditions, [18]. One can try to do the "only if" part in a similar manner. Suppose that $0 < \beta \ne \lambda > 0$. Let $u_n := z_n/\sqrt{\beta\lambda}$, $c := \alpha/(\beta\lambda)$, and $a := \sqrt{\beta/\lambda} \ne 1$. Then the dynamics of (5) can be represented by

$$x_{n+1} = \frac{c + a\,x_n + x_{n-1}/a}{x_{n-2}}, \tag{13}$$

just with two parameters, $c \ge 0$ and $a > 0$. Suppose one can construct, for each pair $(c,a) \in [0,\infty) \times [(0,\infty) \setminus \{1\}]$ and some $\rho = \rho_{c,a} \in (1,\infty)$, a "$\rho$-super-energy" function $g = g_{a,c;\rho}\colon (0,\infty)^3 \to (0,\infty)$ such that g is bounded on each bounded subset of $(0,\infty)^3$ and

$$g\left(u_1, u_2, \frac{c + au_2 + u_1/a}{u_0}\right) \ge \rho g(u_0,u_1,u_2) \quad \text{for all positive } u_0,u_1,u_2. \tag{14}$$

Then, by induction, $g(u_n,u_{1+n},u_{2+n}) \ge \rho^n g(u_0,u_1,u_2) \to \infty$ as $n \to \infty$, for any sequence (u_n) satisfying (13). Therefore, and because g is bounded on each bounded subset of $(0,\infty)^3$, it would follow that the sequence (u_n) is unbounded.

For any pair $(c,a) \in [0,\infty) \times (0,\infty))$ and any $\rho \in (1,\infty)$, there is no "ρ-super-energy" function $g\colon (0,\infty)^3 \to (0,\infty)$. This follows because the point $(u_{a,c},u_{a,c},u_{a,c})$ with $u_{a,c} := \dfrac{1 + a^2 + \sqrt{a^4 + 4a^2c + 2a^2 + 1}}{2a}$ is a fixed point (in fact, the only fixed point) of the map T given by the formula $T(u_0,u_1,u_2) = \left(u_1, u_2, \dfrac{c + au_2 + u_1/a}{u_0}\right)$. If $a \ne 1$, then this point is the only fixed point, [19], of the map T^2 as well. This also disproves, in general, the "only if" part of the conjecture defined by (6). ■

However, One may now try to amend this conjecture by excluding the initial point $(u_{a,c},u_{a,c},u_{a,c})$. Then, accordingly, the definition of a "ρ-super-energy" function would have it defined on a subset (say S) of the set $(0,\infty)^3 \setminus \{(u_{a,c},u_{a,c},u_{a,c})\}$, instead of $(0,\infty)^3$. Such a subset may be allowed to depend on the choice of the initial point (u_0,u_1,u_2), say on its distance from the fixed point $(u_{a,c},u_{a,c},u_{a,c})$. Then one would have also to prove that S is invariant under the map T.

3. Analysis & Discussion

Case 1

Let us try $\alpha = \beta = \gamma$ as positive initial conditions in the dynamic equation (6), the well-known Todd's difference equation which is of the third-order. All its solutions are bounded as illustrated in Figure 1.

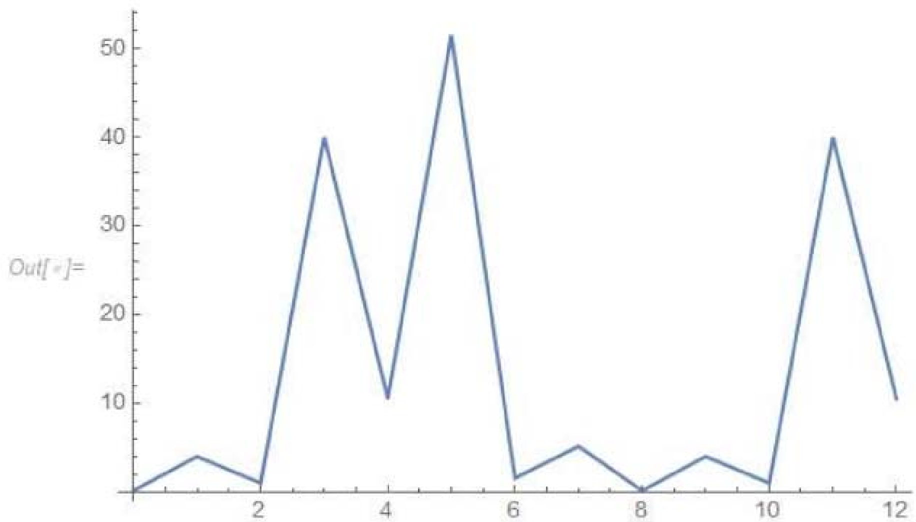

Figure 1: Plot of the bounded solution, versus discretized time, for Todd's dynamic equation in the case when $\alpha = \beta = \gamma$.

The *Mathematica* code, used to generate this solution is:
$(z(0){:=}0.1; z(1){:=}0.1; z(2){:=}0.1;)$
$(\text{Clear}[z];)(\{\alpha = 0.1, \beta = 0.1, \lambda = 12\};)$
$\left(z(\text{n\_}){:=}z(n) = \frac{\alpha+\beta z(n-1)+\lambda z(n-2)}{z(n-3)};\right)$
$\text{ListLinePlot}[\text{Table}[\{n, z(n)\}, \{n, 0, 112\}]].$

Case 2

We try now : $\alpha = \beta < \gamma = 1$. In this case, we still have a bounded solution, as shown in Figure 2. The corresponding *Mathematica* code is:
$(z(0){:=}0.1; z(1){:=}0.1; z(2){:=}0.5;)$
$(\text{Clear}[z];)(\{\alpha = 0.1, \beta = 0.1, \lambda = 12\};)$
$\left(z(\text{n\_}){:=}z(n) = \frac{\alpha+\beta z(n-1)+\lambda z(n-2)}{z(n-3)};\right)$
$\text{ListLinePlot}[\text{Table}[\{n, z(n)\}, \{n, 0, 112\}]].$

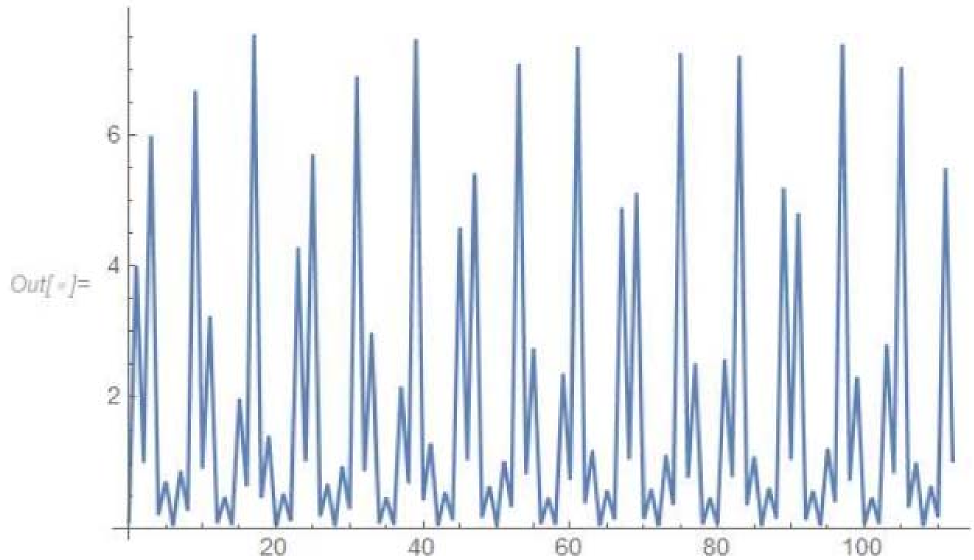

Figure 2: Plot of the bounded solution, versus discretized time, for the dynamic equation (6) in the case when $\alpha = \beta < \gamma = 1$.

Case 3

In this case, we try : $\alpha = \beta < \gamma,\ \gamma > 1$, to obtain also an unbounded solution, as illustrated in Figure 3. The corresponding *Mathematica* code is:

$(z(0){:=}0.1; z(1){:=}0.1; z(2){:=}0.5;)$

$(\text{Clear}[z];)(\{\alpha = 0.1, \beta = 0.1, \lambda = 12\};)$

$\left(z(\text{n\_}){:=}z(n) = \frac{\alpha+\beta z(n-1)+\lambda z(n-2)}{z(n-3)};\right)$

ListLinePlot[Table[{$n, z(n)$}, {$n, 0, 112$}]].

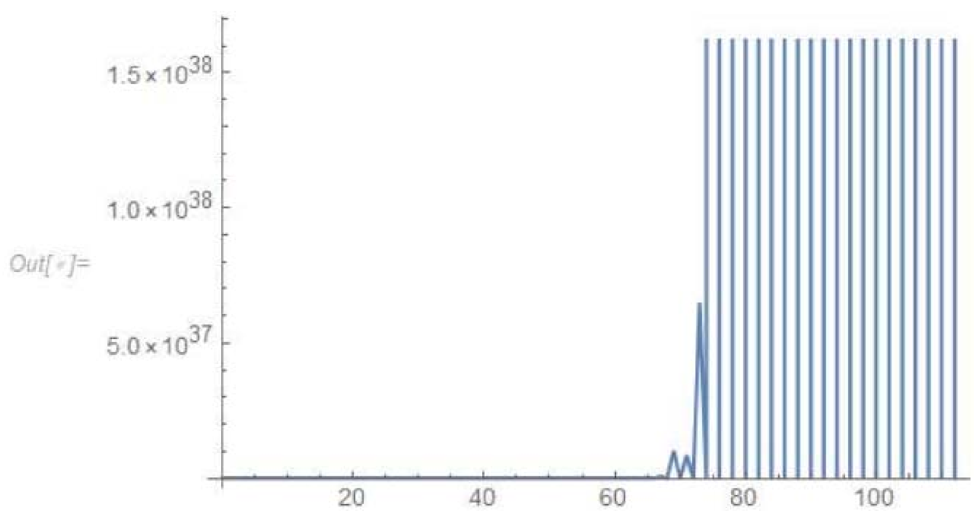

Figure 3: Plot of the bounded solution, versus discretized time, for the dynamic equation (6) in the case when $\alpha = \beta < \gamma,\ \gamma > 1$.

This figure reveals a a good deal about the "only if " part of the considered conjecture. One of the noted indications of these computations is that σ has to lie in the range $(0,1]$ for the conjecture to hold. Moreover, it is desirable to try more values for α and β.

4. Chebyshev Approximation for Bounds

The previous computations demonstrate an interesting behavior of solutions to the dynamic equation (6). In particular, the bounds turn out to behave linearly when using Chebyshev approximation as shown in Figure 4.

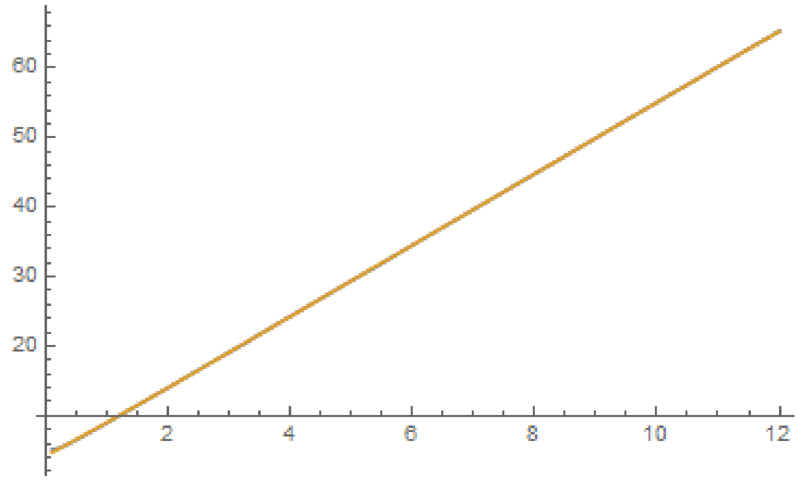

Figure 4: Plot of the Chebyshev approximation bound, versus discretized time, for the solution of the dynamic equation (6).

Here the corresponding *Mathematica* code is:

(Clear[z,λ,ϕ,α];)

({};)(z(α_)(0):=α

;z(α_)(1):=α;z(α_)

;z(α_)(1):=α;z(α_)

$:=z(\alpha)(n)=\frac{\alpha z(\alpha)(n-2)+\alpha z(\alpha)(n-1)+\alpha}{z(\alpha)(n-3)};$

ϕ(α_):=Interpolation[Table[{n,z(α)(n)},{n,0,1112}]]

max(α_):=max(Table[ϕ(α)(n),{n,0,1112}]);

αmin=0.1;αmax=12;δα=0.1;

Lα=Table[{ α,max(α)},{ α,αmin,αmax,δα}];

ϕα=Interpolation[Lα];

poliap(f_,a_,b_,n_)(x_):=Module$\big[$ {zeros, f0, P, ξ, sys, sol, Q}, Null$\big\{$f0(xx_)

$:=f\left(\frac{1}{2}xx(b-a)+\frac{a+b}{2}\right);$

$P(xx\_):=\sum_{j=0}^{n} c(j)T_j(xx);$

$\xi(j\_):=\cos\left(\frac{\pi\left(j-\frac{1}{2}+1\right)}{n+1}\right)+0.;$

sys=(#1=0&)/@Table[f0(ξ(j))-P(ξ(j)),{j,1,n-1}]/. {c(jj_):→c(⌊jj⌋)};

sol=Flatten[Solve[Join[{f0(ξ(n))-P(ξ(n))=0,f0(ξ(0))-P(ξ(0))=0},sys],Table[c(j),{j,0,n}]]];
$Q(\text{xx\_}):=P\left(-\frac{2\text{xx}}{a-b}-\frac{-a-b}{a-b}\right)//.\,\text{sol}\Big\};\text{Collect}[Q(x),x]\Big]\,;$
(degree=5;)
(Chebyshev=poliap(ϕα(#1)&,αmin,αmax,degree)(α))
Plot[Evaluate[{ ϕα(α),poliap(ϕα(#1)&,αmin,αmax,degree)(α)}],{ α,αmin,αmax},PlotRange→All]
The output is :

$$\text{Chebyshev} = -0.0000627537\alpha^5 + 0.00216454\alpha^4 - 0.0283682\alpha^3 + 0.174398\alpha^2 + 4.6394\alpha + 4.27863.$$

5. Conclusion

It is well known that Lyaponov theory, and even some other advanced theories in the realm of differential equations, are unable to provide an affirmative answer to certain challenging problems in rational dynamical systems. This work is a demonstration that a ready affirmative answer to such problems can be found, in physics, using some interpretations that depend on the behavior and properties of high energy functions (like super energy functions) and of their Hamiltonian operator.

Data Availability No data were used to support this paper.

Conflicts of Interest The authors declare no conflicts of interest with regard to any individual or organization.

Acknowledgments
The work of this note was supported in part by the University of Batna 2, Algeria, and by the Universidad Nacional de Colombia. Both authors are grateful to an anonymous referee for his crtical reading of the original typescript. The first author would like to express his deep gratitude to Professor Iosif Pinelis, for improving the quality of this paper, and to members of his family for their moral support.

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Article history: Submitted January, 01, 2021; Revised August, 24, 2022; Accepted September, 02, 2022.