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# On Impulsive Integrodifferential Equations With State-Dependent Delay

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**Abstract**. *In this work, we study the existence of mild solutions for a class of impulsive integrodifferential equations with state-dependent delay. The results are obtained by using Banach contraction principle, Krasnoselskii's and Schaefer's fixed point theorems combined with the theory of resolvent operators in the Grimmer sense. Finally, examples are provided to illustrate the obtained results.*

**Key words**: Integrodifferential Equations, State-Dependent Delay, Mild Solution, Resolvent Operator, Semigroup Theory, Fixed Point Theorems, Impulsive Conditions.

**AMS Subject Classifications**: 34K05, 26A33, 34A12, 35R12, 45J05

## 1. Introduction

Impulsive dynamical systems frequently display piecewise continuous evolutions, which are commonly characterized by ordinary differential equations and instantaneous state jumps, as well as piecewise discontinuous evolutions. Many evolution processes, such as optimal control models in economics, stimulated neural networks, frequency modulated systems, and some motions of missiles or aircraft, are characterized by impulsive dynamical behaviors. It is therefore extremely important to conduct research on impulsive systems. Due to their importance in both theory and practice, the analysis of impulsive systems has recently acquired popularity; see, for example,[1, 2, 3, 4], and the references therein.
Otherwise, the classical features of the semigroups of bounded linear operator are carefully connected to the resolution of differential and integrodifferential systems in Banach spaces. These concepts were used, in the end, to the study of a particularly interesting class of nonlinear integrodifferential equations in Banach spaces. We advise the reader to Pazy [5] for

further inspiration on this topic. Furthermore, functional differential equations with state-dependent delay (SDD) exist often in many applications, such as medical and physical models, and as a result, the investigation of this type of problem has gained a significant deal of attention in recent years. We refer the reader to the handbook of Cañada et al. [6] for more information on the theory of differential equations with state-dependent delay and their applications, as well as the papers [7, 8, 9, 10]. The subject of the existence of solutions for partial functional differential equations with state-dependent delay has recently been addressed in the literature by [11] and [12].

On the other hand, impulsive differential equations with state-dependent delay arise in many areas of applied mathematics. As such, they have been largely studied during the last few decades. The literature related to these equations is very extensive; see for instance [13, 14]. However, in many areas of practical mathematics, impulsive differential equations with state-dependent delay are encountered. As a result, they have received a great deal of attention in recent decades. The literature on these equations is extremely substantial; for example, see [15] and [14]. Benchohra et al. [15] recently researched the existence of impulsive differential equations with state-dependent delay by applying the Burton-Kirk fixed point theorem for the sum of two operators. Authors in [16] demonstrated the existence of mild solutions for partial functional differential equations with state-dependent delay using a nonlinear alternative of the Leray-Schauder type and integrated semigroup theory. Zhu et al. [17], using the theory of $\beta$–resolvant family and fixed point theorems, investigated the local and global existence of mild solutions to a class of nonlinear fractional reaction-diffusion equations with delay. The existence of mild solutions for nonlocal impulsive neutral functional integrodiffential equations with constrained delay was established by the authors in [3] utilizing Hausdorsff's measurements of noncompactness. Using the resolvent operators theory and Darbo's fixed point theorem, Diallo et al., [18], established adequate conditions for the existence of a mild solution for specific impulsive integrodifferential equations in Banach spaces. The existence of solutions for various integrodifferential equations with state-dependent delay and nonlocal conditions in Fréchet spaces was recently examined by Diop et al.[19].

Motivated by the above discussions, our main objective in this work is to investigate the existence of mild solutions for an impulsive integrodifferential equation with state-dependent delay of the form

$$\begin{aligned} &\frac{d}{dt}z(t) = Az(t) + \int_0^t B(t-s)z(s)ds + h\Big(t, z_{\sigma(t,\, z_t)}, (\Im z)(t)\Big), \text{a.e. on, } J \setminus \{t_1, t_2, \dots, t_n\}, \\ &\qquad\qquad + h(t, y_{\rho(t,\, y_t)}),\ \Delta z(t_j) = I_j(z(t_j^-)),\ j = 1, \dots, n, \qquad (1) \\ &z(t) = \varphi(t) \in \mathsf{B}_g, \quad t \in (-\infty, 0], \end{aligned}$$

where $J = [0, a]$, $A : D(A) \subset \mathrm{X} \to \mathrm{X}$ is the infinitesimal generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space X with norm $\|\cdot\|_{\mathrm{X}}$, $(B(t))_{t \geq 0}$ is a family of closed linear operators on X with domain $D(B(t)) \supset D(A)$, $h : J \times \mathsf{B}_g \times \mathrm{X} \to \mathrm{X}$, is a X-valued function. We set $\mathsf{K} = \{(t, s) \in J \times J :\ 0 \leq s \leq t \leq a\}$ and we introduce the non-linear operator $\Im$ given by

$$(\Im z)(t) = \int_0^t v(t, s, z_{\sigma(s,\, z_s)})ds,$$

with $v : \mathsf{K} \times \mathsf{B}_g \to \mathrm{X}$, $\sigma : J \times \mathsf{B}_g \to (-\infty, a]$ some appropriate functions, and $\mathsf{B}$ is a phase space which will be defined later in preliminaries. For $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = a$, the functions $I_j : \mathrm{X} \to \mathrm{X}$ $(j = 1, 2, \ldots, n)$ characterize the jump of the solutions at the impulse points $t_j$, and $z(t_j^+) = \lim_{\alpha\to 0} z(t_j + \alpha)$, $z(t_j^-) = \lim_{\alpha\to 0} z(t_j - \alpha)$ are respectively the right and left limits of $z$ at the points $t_j$.

We denote by $z_t$ the function belonging to $\mathsf{B}_g$ and given by $z_t(\theta) = z(t+\theta)$ for $\theta \leq 0$. Now $z_t(\bullet)$ characterize the history of the state from every $\theta \in (-\infty, 0]$ likely the current time $t$.

The following are the most significant contributions of this article:

(i) A new class of impulsive integrodifferential equations with state-dependent delay in Banach spaces is formulated (see Section 1).

(ii) The key advantage of the targeted technique is that it is based on resolvent operator theory in the sense of Grimmer, Banach fixed point theorem, Schaefer's fixed point Theorem, Krasnoselskii's fixed point Theorem as well as relevant hypotheses (see Section 3).

(iii) Examples are offered in order to validate the theoretical findings that have been proposed (see Section 4).

The remainder of this work is organized as follows: Section 2 presents a number of introductory notions and lemmas that will be utilized to support our primary findings. Following the application of a number of fixed point theorems, we showed that mild solutions to Eq. (1) can be found under a variety of different conditions. Finally, in the last section, two examples are provided to illustrate our obtained results.

## 2. Preliminaries

In this section, we give some definitions and preliminary results from functional analysis and the general theory of integrodifferential equations in Banach spaces. These general notions will be used through this work to prove our main outcomes.

Let X be a Banach space with its norm denoted as $\|\bullet\|_{\mathrm{X}}$ and $L(\mathrm{X})$ represents the Banach space of all bounded linear operators from X into X, and the corresponding norm is denoted by $\|\bullet\|_{L(\mathrm{X})}$. In addition, $F_r(x, \mathrm{X})$ represents the closed ball in X centered in $x$ and of radius $r$.

It should be stated that once the delay is infinite, we should discuss the theoretical phase space $\mathsf{B}_g$ in a convenient way construction. Let $g : (-\infty, 0] \to (0, \infty)$ be a continuous function with $p = \int_{-\infty}^{0} g(t)dt < \infty$. For any $r > 0$, we define

$$\mathsf{B} = \{\xi : [-r, 0] \to \mathrm{X} \text{ such that } \xi(t) \text{ is bounded \& measurable}\}$$

and equip the space $\mathsf{B}$ with the norm

$$\|\xi\|_{[-r,0]} = \sup_{s\in[-r,0]} |\xi(s)|, \ \forall \xi \in \mathsf{B}.$$

Let us define by

$$\mathsf{B}_g = \Big\{\xi : (-\infty, 0] \to \mathrm{X}, \text{ such that for any } q > 0, \ \xi_{[-q,0]} \in \mathsf{B} \\ \text{with } \xi(0) = 0 \text{ and} \int_{-\infty}^{0} g(s)\|\xi\|_{[s,0]}ds < \infty\Big\}.$$

If $\mathsf{B}_g$ is endowed with the norm

$$\|\xi\|_{\mathsf{B}_g} = \int_{-\infty}^{0} g(s)\|\xi\|_{[s,0]}ds, \ \forall\ \xi \in \mathsf{B}_g,$$

then it is known that $(\mathsf{B}_g, \|\cdot\|_{\mathsf{B}_g})$ is a Banach space.
Now, we consider the space

$$\mathsf{B}^{'} = \Big\{z : (-\infty, a] \to \mathrm{X} \text{ such that } z|_{J_j} \in C(J_j, \mathrm{X}) z(t_j^+) \ \&\ \exists\ z(t_j^+) \text{ and } z(t_j^-) \\ \text{with } z(t_j) = z(t_j^-),\ z_0 = \varphi \in \mathsf{B}_g,\ j = 1, 2, \ldots, n\Big\},$$

where $z|_{J_j}$ is the restriction of $z$ to $J_j = (t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots, n$. Let $\|\cdot\|_{\mathsf{B}'}$ be the seminorm in $\mathsf{B}^{'}$ defined by

$$\|z\|_{\mathsf{B}'} = \sup\{|z(s)|: s \in [0, a]\} + \|z\|_{\mathsf{B}'}, \quad z \in \mathsf{B}^{'}.$$

If $z : (-\infty, a] \to \mathrm{X}$, $\ a > 0$, is continuous on $J$ and $z_0 \in \mathsf{B}_g$, then for all $t \in J$, the following conditions hold:
(**A**) $z_t$ is in $\mathsf{B}_g$;
(**B**) $\|z(t)\|_{\mathrm{X}} \le H\|z_t\|_{\mathsf{B}_g}$;
(**C**)$\|z_t\|_{\mathsf{B}_g} \le K_1(t)\sup\{\|z(s)\| \ : \ 0 \le s \le t\} + K_2(t)\|z_0\|_{\mathsf{B}_g}$ where $H > 0$ is a constant and $K_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $K_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and $K_1$, $K_2$ are independent of $z(\cdot)$.
(**D**)The function $t \to \varphi_t$ is well described and continuous from the set

$$\mathsf{R}(\sigma^-) = \{\sigma(s, \psi) : (s, \psi) \in [0, a] \times \mathsf{B}_g\},$$

into $\mathsf{B}_g$ and there is a continuous and bounded function $J^\varphi : \mathsf{R}(\sigma^-) \to (0, \infty)$ to ensure that $\|\varphi_t\|_{\mathsf{B}_g} \le J^\varphi(t)\|\varphi\|_{\mathsf{B}_g}$ for every $t \in \mathsf{R}(\sigma^-)$.

We give now the following estimate which is very useful.

**Lemma 2.1** [20, Lemma 2.1]. *Let* $z : (-\infty, a] \to X$ *be a function in a way that* $z_0 = \varphi$, $z|_{J_j} \in C(J_j, X)$ *and if (D) holds, then*

$$\|z_s\|_{\mathsf{B}_g} \le (K_2^* + J^\varphi)\|\varphi\|_{\mathsf{B}_g} + K_1^* \sup\{\|z(\theta)\|_{\mathrm{X}} : \theta \in [0, \max\{0, s\}]\},\ s \in \mathsf{R}(\sigma^-) \cup J,$$

*where* $J^\varphi = \sup_{t \in \mathsf{R}(\sigma^-)} J^\varphi(t)$, $K_1^* = \sup_{s \in [0,a]} K_1(s)$, $K_2^* = \sup_{s \in [0,a]} K_2(s)$.

Next we include the following classical fixed point theorems which will be used in the sequel to prove our results.

**Theorem 2.1** [21], (Banach). *Any contraction mapping of a complete non-empty metric space E into E has a unique fixed point in E.*

**Theorem 2.2** [21], (Schaefer). *Let* $S : X \to X$ *be a completely continuous mapping. Then either*

1. *S has a fixed point, or*
2. *the set* $\{x \in X : x = \alpha S(x),\ \alpha \in (0,1)\}$ *is unbounded.*

**Theorem 2.3** [**21**], (Krasnoselskii). *Let D be a closed, convex, and non-empty subset of a Banach space X. Suppose that F and G map D into X such that*
*(i)* $Fx + Gy \in D$, *for all* $x, y \in D$;
*(ii) F is compact and continuous;*
*(iii) G is a contraction mapping.*
*Then, there exists* $z \in D$ *such that* $z = Fz + Gz$.

Next, to be able to access existence of mild solution for Eq. (1), we introduce partial integrodifferential equations and resolvent operators that are used to develop the main results of this work.
Let $\mathsf{D}$ be Banach space . We denote by $\mathsf{L}(\mathsf{D},\mathsf{Y})$ the Banach space of bounded linear operators from $\mathsf{D}$ into $\mathsf{Y}$ endowed with operator norm and we abbreviate this notation to $\mathsf{L}(\mathsf{D})$ when $\mathsf{D} = \mathsf{Y}$.
In what follows, $A$ and $B(t)$ are closed linear operators on $\mathsf{D}$. $\mathsf{Y}$ represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_{\mathsf{Y}} := |Ay|_{\mathrm{X}} + |y|_{\mathrm{X}} \qquad \text{for} \quad y \in \mathsf{Y}.$$

The notations $C([0,+\infty);\mathsf{Y})$, $\mathsf{L}(\mathsf{Y},\mathsf{D})$ stand for the space of all continuous functions from $[0,+\infty)$ into $\mathsf{Y}$, the set of all bounded linear operators from $\mathsf{Y}$ into $\mathsf{D}$, respectively.
Assume that

(**H1**) $A$ is the infinitesimal generator of $C_0$-semigroup $(T(t))_{t \geq 0}$ in $\mathsf{D}$.

(**H2**) $(B(t))_{t \in \mathsf{I}}$ is a family of closed linear operator on $\mathsf{Y}$ such that $B(t)$ is continuous when regarded as linear map from to $(\mathsf{Y}, |\cdot|)$ into $(\mathsf{D}, \|\cdot\|_{\mathsf{D}})$ and the map $t \to B(t)y$ is measurable for all $y \in \mathsf{Y}$ and $t \in \mathsf{I}$ and belong to $\mathsf{W}^{1,1}(\mathsf{I},\mathsf{D})$. Moreover there exists an integrable function $c : [0,+\infty[ \to \mathbb{R}^+$ such as

$$\left\| \frac{d}{dt} B(t)y \right\| \leq c(t)\|y\|_{\mathsf{Y}}, \quad y \in \mathsf{Y}, t \in \mathsf{I}.$$

Now, we consider the following Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds \quad \text{for } t \geq 0 \\ y(0) = y_0 \in \mathrm{X}. \end{cases} \tag{2}$$

**Definition 2.1** [22]. We call resolvent operator for Eq. (2), a bounded linear operator valued function $R(t) \in \mathsf{L}(\mathsf{D})$ for $t \geq 0$, verifying the following properties:
(i) $R(0) = I$ (identity operator on $\mathsf{D}$) and $\|R(t)\| \leq Me^{\omega t}$ for some constants $M$ and $\omega$.
(ii) For each $x \in \mathrm{X}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(iii) $R(t) \in \mathsf{L}(\mathsf{Y})$ for $t \geq 0$. For $x \in \mathsf{Y}$, $R(\cdot)x \in C^1([0,+\infty),\mathsf{D}) \cap C([0,+\infty),\mathsf{Y})$ and

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds$$
$$= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad \text{for} \quad t \geq 0.$$

**Theorem 2.4** [23]. *Suppose that (H1) and (H2) hold. Then, Eq. (2) has a unique resolvent operator* $(R(t))_{t \geq 0}$.

Then we give the following important estimate.

**Lemma 2.2** [24]. *Let (H1) and (H2) be satisfied. Then for all* $t > 0$ *there exists a constant* $\varpi$ *such that*

$$\|R(t+\tau) - R(\tau)R(t)\|_{L(X)} \leq \varpi\tau \text{ for } 0 \leq \tau \leq t \leq a.$$

**Lemma 2.3** [25]. *Assume that (H1) and (H2) hold. The resolvent operator* $(R(t))_{t\geq 0}$ *is compact for* $t > 0$ *if only if the semigroup* $(T(t))_{t\geq 0}$ *is compact for* $t > 0$.

In the following, we give some results for the existence of solutions for the following integrodifferential equation.

$$\begin{cases} y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds + q(t) \quad \text{for } t \geq 0, \\ y(0) = y_0 \in \mathsf{D}, \end{cases} \tag{3}$$

where $q : [0,+\infty) \to \mathsf{D}$ is a continuous function.

**Definition 2.2** [23]. A continuous function $y : [0,+\infty) \to \mathsf{D}$ is said to be a strict solution of Eq. (3) if

1. $y \in C^1([0,+\infty), \mathsf{D}) \cap C([0,+\infty), \mathsf{Y})$,
2. $y$ satisfies Eq. (3) for $t \geq 0$.

**Theorem 2.5** [23]. *Assume that hypotheses (H1) and (H2) hold. If y is a strict solution of Eq. (3), then the variation of constant formula holds, i.e.*

$$y(t) = R(t)y_0 + \int_0^t R(t-s)q(s)ds \quad \textit{for } t \geq 0. \tag{4}$$

In accordance with the above discussion, we define the mild solution of Eq. (1) as follows.

**Definition 2.3**. A function $z : (-\infty, a] \to X$ is said to be a mild solution of Eq. (1) if: $z_0 = \varphi \in \mathsf{B}_g$ on $(-\infty, 0]$; $\Delta z|_{t=t_n} = I_j(z(t_j^-))$, $j = 1, 2, \ldots, n$, the restriction of $z(\cdot)$ to the interval $[0,a]\backslash\{t_1, t_2, \ldots, t_n\}$ is continuous and satisfies the following integral equation:

$$z(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ R(t)\varphi(0) + \int_0^t R(t-s)\, h\Big(s, z_{\sigma(s,\, z_s)}, \int_0^s v(s,\tau, z_{\sigma(\tau,\, z_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\ R(t)\varphi(0) + R(t-t_1)\, I_1(z(t_1^-)) & \\ \qquad + \int_0^t R(t-s)\, h\Big(s, z_{\sigma(s,\, z_s)}, \int_0^s v(s,\tau, z_{\sigma(\tau,\, z_\tau)})d\tau\Big)ds, & t \in (t_1, t_2], \\ \qquad \vdots & \\ R(t)\varphi(0) + \sum_{j=1}^{n} R(t-t_j)\, I_j(z(t_j^-)) & \\ \qquad + \int_0^t R(t-s)\, h\Big(s, z_{\sigma(s,\, z_s)}, \int_0^s v(s,\tau, {}_{\sigma(\tau,\, z_\tau)})d\tau\Big)ds, & t \in (t_n, a]. \end{cases} \tag{5}$$

Let $M = \sup_{t\in[0,\, a]} \|R(t)\|$ to adopt the following hypotheses.

**(H3)** $(R(t))_{t\geq 0}$ is compact for $t > 0$.

**(H4)** $h : J \times \mathsf{B}_g \times \mathrm{X} \to \mathrm{X}$ is continuous and there exist some functions $\gamma_1, \gamma_2 \in L(J, [0, +\infty))$ such that

$$\|h(t,\varphi,x) - h(t,\psi,y)\|_{\mathrm{X}} \leq \gamma_1(t)\|\varphi-\psi\|_{\mathsf{B}_g} + \gamma_2(t)\|x-y\|_{\mathrm{X}},\ t \in J,\ (\varphi,\psi) \in \mathsf{B}_g^2,\ x, y \in \mathrm{X}.$$

**(H5)** $v : \mathsf{K} \times \mathsf{B}_g \to \mathrm{X}$ is continuous and there is a function $\delta_1 \in C(J, [0, +\infty))$ such that

$$\|v(t,s,\varphi) - v(t,s,\psi)\|_{\mathrm{X}} \leq \delta_1(t)\|\varphi-\psi\|_{\mathsf{B}_g},\ \ (t,s) \in \mathsf{K},\ \ (\varphi,\psi) \in \mathsf{B}_g^2.$$

**(H6)** For each $j = 1, 2, \ldots, n,$ there exist $D_j \in C(J, [0, +\infty))$ such that

$$\|I_j(x) - I_j(y)\|_{\mathrm{X}} \leq D_j(t)\|x-y\|_{\mathrm{X}},\ \text{for all}\ \ x, y \in \mathrm{X}.$$

**(H7)** $\Lambda_n(a) = \Big[ M\mathsf{K}_1^* \int_0^a [\gamma_1(s) + a\,\delta_1^0\,\gamma_2(s)]ds + nMD_0 \Big] < 1,$ where $\delta_1^0 = \max\{\delta_1(t) | t \in J\}$ and $D_0 = \max\{D_j(t) | t \in J,\ j = 1, 2, \ldots, n\}$.

## 3. Existence of the Mild Solutions for Eq. (1)

Taking into account the above notations, definitions and lemmas, we shall derive the existence of solutions for Eq. (1) by using the contraction mapping principle, Krasnoselskii fixed point theorem and Schaefer's fixed point theorem.

**Theorem 3.1**. *Assume that (H1), (H2) and (H4)-(H7) are satisfied. Then, Eq. (1) has a unique mild solution on* $(-\infty, a]$.

*Proof.* Define the operator $\Phi : \mathsf{B}' \to \mathsf{B}'$ by

$$(\Phi z)(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ R(t)\varphi(0) + \int_0^t R(t-s)h\Big(s, z_{\sigma(s,z_s)},\ \int_0^s v(s,\tau,z_{\sigma(\tau,z_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\ R(t)\varphi(0) + R(t-t_1)I_1(z(t_1^-)) & \\ \quad + \int_0^t R(t-s)h\Big(s, z_{\sigma(s,z_s)}, \int_0^s v(s,\tau,z_{\sigma(\tau,z_\tau)})d\tau\Big)ds, & t \in (t_1, t_2], \\ \vdots & \\ R(t)\varphi(0) + \sum_{j=1}^n R(t-t_j)I_j(z(t_j^-)) & \\ \quad + \int_0^t R(t-s)h\Big(s, z_{\sigma(s,z_s)}, \int_0^s v(s,\tau,z_{\sigma(\tau,z_\tau)})d\tau\Big)ds, & t \in (t_n, a]. \end{cases}$$

It is obvious that the fixed points of the operator $\Phi$ are mild solutions of Eq. (1). Then we define the function $x(\cdot)\ : (-\infty, a] \to X$ by

$$x(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ R(t)\varphi(0), & t \in J, \end{cases}$$

then $x_0 = \varphi$. For every function $f \in C(J, \mathbb{R})$ with $f(0) = 0$, we introduce $\bar{f}$ given by

$$\bar{f}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ f(t), & t \in J. \end{cases}$$

If $z(\cdot)$ verifies (5), we can decompose it as $z(t) = x(t) + \bar{f}(t),\ t \in J$. It follows that $z_t = x_t + \bar{f}_t$, for each $t \in J$ and the function $f(\cdot)$ satisfies

$$f(t) = \begin{cases} \int_0^t R(t-s)h\ \Big(s, \bar{f}_{\sigma(s,\bar{f}_s + x_s)} + x_{\sigma(s,\bar{f}_s + x_s)}, & \\ \qquad \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau + x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau + x_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\ R(t-t_1)I_1(x(t_1^-) + \bar{f}(t_1^-))\ + \int_0^t R(t-s)\ h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s + x_s)} + x_{\sigma(s,\bar{f}_s + x_s)}, & \\ \qquad \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau + x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau + x_\tau)})d\tau\Big)\ ds, & t \in (t_1, t_2], \\ \vdots & \\ \sum_{j=1}^n R(t-t_j)\ I_j(x(t_j^-) + \bar{f}(t_j^-))\ + \int_0^t R(t-s)\ h\ \Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s + x_s)} + x_{\sigma(s,\bar{f}_s + x_s)}, & \\ \qquad \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau + x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau + x_\tau)})d\tau\Big)ds, & t \in (t_n, a]. \end{cases}$$

Let $\mathcal{B}'' = \{f \in \mathcal{B}' : f_0 = 0\}$. Let $\|\cdot\|_{\mathcal{B}''}$ be the norm in $\mathcal{B}''$ defined by

$$\|f\|_{\mathcal{B}''} = \sup_{t \in J} \|f(t)\|_X, \quad f \in \mathcal{B}''.$$

$(\mathcal{B}'', \|\cdot\|_{\mathcal{B}''})$ is a Banach space.

We introduce now the operator $\overline{\Phi} : \mathcal{B}'' \to \mathcal{B}''$ by

$$(\overline{\Phi}f)(t) = \begin{cases} \int_0^t R(t-s)\, h\Big(s, \overline{f}_{\sigma(s,\overline{f}_s+x_s)} + x_{\sigma(s,\overline{f}_s+x_s)}, \\ \qquad \int_0^s v(s,\tau, \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} + x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\Big)ds, \quad t \in [0,t_1], \\ R(t-t_1)\, I_1(x(t_1^-)+\overline{f}(t_1^-)) + \int_0^t R(t-s)\, h\Big(s, \overline{f}_{\sigma(s,\overline{f}_s+x_s)} + x_{\sigma(s,\overline{f}_s+x_s)}, \\ \qquad \int_0^s v(s,\tau, \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} + x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\Big)ds, \quad t \in (t_1,t_2], \\ \vdots \\ \sum_{j=1}^n R(t-t_j)\, I_j(x(t_j^-)+\overline{f}(t_j^-)) + \int_0^t R(t-s)\, h\Big(s, \overline{f}_{\sigma(s,\overline{f}_s+x_s)} + x_{\sigma(s,\overline{f}_s+x_s)}, \\ \qquad \int_0^s v(s,\tau, \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} + x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\Big)ds, \quad t \in (t_n,a]. \end{cases}$$

It is obvious that the operator $\Phi$ admits a fixed point if only if $\overline{\Phi}$ has a fixed point. Thus, we are going to prove that $\overline{\Phi}$ admits a fixed point.

**Remark 3.1**. From Lemma 2.1 with the hypotheses (**H4**)-(**H7**), we have the following estimates:

$$\begin{aligned} &\Big\| h\Big(s, \overline{f}_{\sigma(s,\overline{f}_s+x_s)} + x_{\sigma(s,\overline{f}_s+x_s)}, \int_0^s v(s,\tau, \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} + x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\Big) \\ &-h\Big(s, \overline{f}^*_{\sigma(s,\overline{f}_s+x_s)} + x_{\sigma(s,\overline{f}_s+x_s)}, \int_0^s v(s,\tau, \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} + x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\Big)\Big\|_X \\ &\le \gamma_1(s)\| \overline{f}_{\sigma(s,\overline{f}_s+x_s)} + x_{\sigma(s,\overline{f}_s+x_s)}, -\overline{f}^*_{\sigma(s,\overline{f}_s+x_s)} - x_{\sigma(s,\overline{f}_s+x_s)}\|_{\mathcal{B}_g} \\ &+a\,\delta_1^0\,\gamma_2(s)\| \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} + x_{\sigma(\tau,\overline{f}_\tau+x_\tau)} - \overline{f}^*_{\sigma(\tau,\overline{f}_\tau+x_\tau)} - x_{\sigma(\tau,\overline{f}_\tau+x_\tau)}\|_{\mathcal{B}_g} \\ &\le \gamma_1(s)\|\overline{f}_{\sigma(s,\overline{f}_s+x_s)} - \overline{f}^*_{\sigma(s,\overline{f}_s+x_s)}\|_{\mathcal{B}_g} + a\,\delta_1^0\,\gamma_2(s)\| \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)} - \overline{f}^*_{\sigma(\tau,\overline{f}_\tau+x_\tau)}\|_{\mathcal{B}_g}. \end{aligned} \tag{6}$$

Observe that

$$\begin{aligned} \|\overline{f}_{\sigma(s,\overline{f}_s+x_s)} - \overline{f}^*_{\sigma(s,\overline{f}_s+x_s)}\|_{\mathcal{B}_g} &\le \mathcal{K}_1^* \sup_{0\le\tau\le\sigma(s,\overline{f}_s+x_s)} \|\overline{f}(\tau) - \overline{f}^*(\tau)\|_{\mathcal{B}_g} + \mathcal{K}_2^*\|\overline{f}_0 - \overline{f}_0^*\|_{\mathcal{B}_g} \\ &\le \mathcal{K}_1^* \sup_{0\le\tau\le s} \|\overline{f}(\tau) - \overline{f}^*(\tau)\|_X \\ &\le \mathcal{K}_1^*\|f - f^*\|_{\mathcal{B}''}. \end{aligned}$$

Then (6) becomes

$$\begin{aligned}&\Big\| h\Big(s, \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau, \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\\ &-h\Big(s,\bar{f}^*_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau, \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_X \\ &\leq \mathsf{K}_1^* \gamma_1(s) \| f-f^*\|_{\mathsf{B}''} + \mathsf{K}_1^* \, a\, \delta_1^0 \, \gamma_2(s) \| f-f^*\|_{\mathsf{B}''}\\ &\leq \mathsf{K}_1^*[\gamma_1(s) + a\,\delta_1^0\,\gamma_2(s)]\| f-f^*\|_{\mathsf{B}''}.\end{aligned} \tag{7}$$

Adoption of hypothesis (**H6**), allows writing

$$\begin{aligned}&\left\| \sum_{j=1}^{k} R(t-t_j)I_j(\bar{f}(t_j^-)) - \sum_{j=1}^{k} R(t-t_j)I_j(\bar{f}^*(t_j^-))\right\|_{\mathrm{X}}\\ &\leq \sum_{j=1}^{k} \|R(t-t_j)\|_{L(\mathrm{X})}\|I_j(\bar{f}(t_j^-)) - I_j(\bar{f}^*(t_j^-))\|_{\mathrm{X}}\\ &\leq M\,k\,D_0\| f-f^*\|_{\mathsf{B}''}.\end{aligned} \tag{8}$$

Indeed $f,\ f^* \in \mathsf{B}''$, then from estimate (7), for all $t \in [0,t_1]$, there holds

$$\begin{aligned}&\|\overline{\Phi}(f)(t) - \overline{\Phi}(f^*)(t)\|_{\mathrm{X}}\\ &\quad\leq \int_0^t \|R(t-s)\|_{L(\mathrm{X})}\Big\| h\Big(s,\bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\\ &\quad - h\Big(s,\bar{f}^*_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}^*_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_{\mathrm{X}}\\ &\quad\leq M\,\mathsf{K}_1^* \int_0^t [\gamma_1(s) + a\,\delta_1^0\,\gamma_2(s)]ds\, \| f-f^*\|_{\mathsf{B}''}.\end{aligned}$$

In general, for each $t \in (t_j,t_{j+1}],\ 1 \leq j \leq n$, using the relations (7) and (8), we have

$$\begin{aligned}\|\overline{\Phi}(f)(t) - \overline{\Phi}(f^*)(t)\|_{\mathrm{X}} &\leq MjD_0\|f-f^*\|_{\mathsf{B}''} + M\,\mathsf{K}_1^* \int_0^t [\gamma_1(s) + a\,\delta_1^0\,\gamma_2(s)]ds\|f-f^*\|_{\mathsf{B}''}\\ &\leq \Big(Mj\,D_0 + M\,\mathsf{K}_1^* \int_0^t [\gamma_1(s) + a\,\delta_1^0\,\gamma_2(s)]ds\Big)\|f-f^*\|_{\mathsf{B}''}\\ &\leq \Lambda_j(t)\|f-f^*\|_{\mathsf{B}''}.\end{aligned}$$

When $j = n$, we have

$$\|\overline{\Phi}(f)(t) - \overline{\Phi}(f^*)(t)\|_{\mathrm{X}} \leq nM\,D_0 + M\,\mathsf{K}_1^* \int_0^a [\gamma_1(s) + a\,\delta_1^0\,\gamma_2(s)]ds\| f-f^*\|_{\mathsf{B}''}.$$

Then for all $t \in [0,a]$, we obtain

$$\|\overline{\Phi}(f)(t) - \overline{\Phi}(f^*)(t)\|_{\mathsf{B}''} \leq \Lambda_n(a)\| f - f^*\|_{\mathsf{B}''}.$$

Since $\Lambda_j(t) \leq \Lambda_n(a)$, with the hypothesis (**H7**) and in the perspective of the contraction mapping principle, we conclude that $\overline{\Phi}$ has a unique fixed point $f \in \mathsf{B}''$ which is a mild solution of Eq. (1) on $(-\infty, a]$. ■

Now, we prove another existence result of mild solution for Eq. (1). For that the following hypotheses are needed.
(**H8**) The function $h : J \times \mathsf{B}_g \times \mathrm{X} \to \mathrm{X}$ is continuous and there exist some functions $\gamma_3,\ \gamma_4,\ \gamma_5$ in $C(J, \mathbb{R}^+)$ such that

$$\|h(t,\varphi,z)\|_{\mathrm{X}} \leq \gamma_3(t) + \gamma_4(t)\|\varphi\|_{\mathsf{B}_g} + \gamma_5\|z\|_{\mathrm{X}}, \quad (t,\varphi,z) \in J \times \mathsf{B}_g \times \mathrm{X}.$$

(**H9**) The function $v : \mathsf{K} \times \mathsf{B}_g \to \mathrm{X}$ is continuous and there exist functions $\delta_2, \delta_3 \in C(J, \mathbb{R}^+)$ such that

$$\|v(t,s,\psi)\|_{\mathrm{X}} \leq \delta_2(s) + \delta_3(s)\|\psi\|_{\mathsf{B}_g}, \quad \psi \in \mathsf{B}_g.$$

(**H10**) For each $j = 1, 2, \ldots, n,$ there exist $D_j \in C(J, \mathbb{R}^+)$ such that

$$\|I_j(z)\|_{\mathrm{X}} \leq D_j(t)\|z\|_{\mathrm{X}}, \quad t \in J.$$

**Remark 3.2**. From Lemma 2.1 and hypotheses (**H8**)-(**H10**), we have the following estimates:
(**e1**)

$$\begin{aligned}
&\|\bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}\|_{\mathsf{B}_g} \\
&\quad \leq \|\bar{f}_{\sigma(s,\bar{f}_s+x_s)}\|_{\mathsf{B}_g} + \|x_{\sigma(s,\bar{f}_s+x_s)}\|_{\mathsf{B}_g} \\
&\quad \leq \mathsf{K}_1^* \sup_{0\leq\tau\leq s}\|f(\tau)\|_{\mathrm{X}} + (\mathsf{K}_2^* + J^{\varphi})\|f_0\|_{\mathsf{B}_g} + \mathsf{K}_1^*|x(s)| + (\mathsf{K}_2^* + J^{\varphi})\|x_0\|_{\mathsf{B}_g} \\
&\quad \leq \mathsf{K}_1^* \sup_{0\leq\tau\leq s}\|f(\tau)\|_{\mathrm{X}} + \mathsf{K}_1^*\|R(t)\|_{L(\mathrm{X})}|\varphi(0)| + (\mathsf{K}_2^* + J^{\varphi})\|\varphi\|_{\mathsf{B}_g} \\
&\quad \leq \mathsf{K}_1^* \sup_{0\leq\tau\leq s}\|f(\tau)\|_{\mathrm{X}} + \mathsf{K}_1^* MH\|\varphi\|_{\mathsf{B}_g} + (\mathsf{K}_2^* + J^{\varphi})\|\varphi\|_{\mathsf{B}_g} \\
&\quad \leq \mathsf{K}_1^* \sup_{0\leq\tau\leq s}\|f(\tau)\|_{\mathrm{X}} + (\mathsf{K}_1^* MH + \mathsf{K}_2^* + J^{\varphi})\|\varphi\|_{\mathsf{B}_g}.
\end{aligned}$$

Since $\|f\|_{\mathrm{X}} < r,\ r > 0,$ then

$$\|\bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}\|_{\mathsf{B}_g} \leq \mathsf{K}_1^* r + c_n,$$

where $c_n = (\mathsf{K}_1^* MH + \mathsf{K}_2^* + J^{\varphi})\|\varphi\|_{\mathsf{B}}$.
(**e2**)

$$
\begin{aligned}
&\int_0^s \|v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})\|_{\mathrm{X}} d\tau \\
&\quad \le \int_0^s (\delta_2(\tau) + \delta_3(\tau)\|\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)}\|_{\mathsf{B}_g}) d\tau \\
&\quad \le \int_0^s \left(\delta_2(\tau) + \delta_3(\tau)\left\{\mathsf{K}_1^* \sup_{0\le\theta\le\tau} \|f(\theta)\|_{\mathrm{X}} + c_n\right\}\right) d\tau \\
&\quad \le \int_0^s \delta_2(\tau)d\tau + c_n \int_0^s \delta_3(\tau)d\tau + \mathsf{K}_1^* \int_0^s \delta_3(\tau) \sup_{0\le\theta\le\tau} \|f(\theta)\|_{\mathrm{X}} d\tau.
\end{aligned}
$$

**(e3)**

$$
\begin{aligned}
&\left\| h\left(s,\bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\right) \right\|_{\mathrm{X}} \\
&\le \gamma_3(s) + \gamma_4(s)\|\bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}\|_{\mathsf{B}_g} + \gamma_5(s)\int_0^s \|v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})\|_{\mathrm{X}} d\tau \\
&\le \gamma_3(s) + \gamma_4(s)\left[\mathsf{K}_1^* \sup_{0\le\tau\le t}\|f(\tau)\|_{\mathrm{X}} + c_n\right] + \gamma_5(s)\left[\int_0^s \delta_2(\tau)d\tau + c_n\int_0^s \delta_3(\tau)d\tau\right. \\
&\left. + \mathsf{K}_1^* \int_0^s \delta_3(\tau) \sup_{0\le\theta\le\tau}\|f(\theta)\|_{\mathrm{X}} d\tau\right] \\
&= \varphi_1(s) + \varphi_2(s) r = \lambda(s),
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_1(s) &= \gamma_3(s) + \gamma_5(s)\int_0^t \delta_2(s)ds + c_n\left(\gamma_4(s) + \gamma_5(s)\int_0^t \delta_3(s)ds\right), \\
\varphi_2(s) &= \mathsf{K}_1^*\gamma_4(s) + \mathsf{K}_1^*\gamma_5(s)\int_0^t \delta_3(s)ds.
\end{aligned}
$$

**(e4)**

$$
\left\|\sum_{j=1}^n R(t-t_j)I_j(x(t_j^-) + \bar{f}(t_j^-))\right\|_{\mathrm{X}} \le nM\|I_j(x(t_j^-) + \bar{f}(t_j^-))\|_{\mathrm{X}}. \tag{9}
$$

Since

$$
\begin{aligned}
|I_j(x(t_j^-)+\bar{f}(t_j^-))| &\le D_j(t)(|x(t_j^-)+\bar{f}(t_j^-)|)\\
&\le D_j(t)\left(\sup_{t\in J}|x(t)+\bar{f}(t)|\right)\\
&\le D_0 H\|x_t+\bar{f}_t\|_{\mathcal{B}_g},
\end{aligned}
$$

where $D_0 = \max\{D_j(t)|t \in J,\ j = 1,2,\dots,n\}$.
We then have:

$$
\begin{aligned}
\|x_t+\bar{f}_t\|_{\mathcal{B}_g} &\le \|x_t\|_{\mathcal{B}_g} + \|\bar{f}_t\|_{\mathcal{B}_g}\\
&\le \mathsf{K}_1(t)\sup_{0\le\tau\le t}\|x(\tau)\|_X + \mathsf{K}_2(t)\|x_0\|_{\mathcal{B}_g} + \mathsf{K}_1(t)\sup_{0\le\tau\le t}\|f(\tau)\|_X + \mathsf{K}_2(t)\|f_0\|_{\mathcal{B}_g}\\
&\le \mathsf{K}_1(t)[\|R(t)\|_{L(X)}|\varphi(0)|] + \mathsf{K}_2(t)\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_1(t)\sup_{0\le\tau\le t}\|f(\tau)\|_X\\
&\le \mathsf{K}_1^* MH\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_1^* r\\
&\le \mathsf{K}_1^*(MH\|\varphi\|_{\mathcal{B}_g} + r) + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g}.
\end{aligned}
$$

Hence, Eq. (9) becomes

$$
\begin{aligned}
\left\|\sum_{j=1}^{n} R(t-t_j)I_j(x(t_j^-)+\bar{f}(t_j^-))\right\|_X &\le nMD_0H\,[\mathsf{K}_1^* MH\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g}] + nMD_0H\mathsf{K}_1^* r\\
&\le N + nMD_0H\mathsf{K}_1^* r,
\end{aligned}
$$

where $N = nMD_0H[\mathsf{K}_1^* MH\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g}]$.
**(e5)**

$$
\begin{aligned}
&\Big\| h\left(s,\ \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\right)\\
&-h\left(s,\bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\right)\Big\|_X\\
&\le 2\left(\gamma_3(s)+\gamma_5(s)\int_0^t \delta_2(s)ds\right) + 2c_n\left(\gamma_4(s)+\gamma_5(s)\int_0^t \delta_3(s)ds\right)\\
&+ 2\mathsf{K}_1^* r\left(\gamma_4(s)+\gamma_5(s)\int_0^t \delta_3(s)ds\right)\\
&\le 2\left(\gamma_3(s)+\gamma_5(s)\int_0^t \delta_2(s)ds\right) + 2(c_n+\mathsf{K}_1^* r)\left(\gamma_4(s)+\gamma_5(s)\int_0^t \delta_3(s)ds\right).
\end{aligned}
$$

**Theorem 3.2**. *Assume that (H1)-(H4) and (H8)-(H10) hold. Then, Eq. (1) has at least one mild solution on* $(-\infty,a]$ *provided that*

$$\mathsf{K}_1^* M \int_0^a [\gamma_1(s) + a\delta_1^0 \gamma_2(s)] ds < 1.$$

*Proof.* We choose $r \geq N + M\int_0^a \varphi_1(s)ds + \left(M\int_0^a \varphi_2(s)ds + nMD_0\mathsf{K}_1^*\right)r$ and consider $F_r = \{f \in \mathsf{B}'' : \|f\|_{\mathsf{B}}'' \leq r\}$, then $F_r$ is a bounded, closed-convex subset in $\mathsf{B}''$.
Let $\Phi_1 : F_r \to F_r$ and $\Phi_2 : F_r \to F_r$ be defined as

$$(\Phi_1 f)(t) = \begin{cases} 0, & t \in [0, t_1], \\ R(t-t_1)I_1(x(t_1^-) + f(t_1^-)), & t \in (t_1, t_2], \\ \vdots & \\ \sum_{j=1}^{n} R(t-t_j)I_j(x(t_j^-) + f(t_j^-)), & t \in (t_n, a], \end{cases} \tag{10}$$

and

$$(\Phi_2 f)(t) = \int_0^t R(t-s)h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)ds,\ t \in J.$$

The proof is divided into the following five steps.

**Step 1**. For $f, f^* \in F_r$, $\Phi_1 f + \Phi_2 f^* \in F_r$.
By Remark 3.2, for $t \in [0, t_1]$, we have

$$\begin{aligned} &\|(\Phi_1 f)(t) + (\Phi_2 f^*)(t)\|_{\mathrm{X}} \\ &\leq \int_0^t \|R(t-s)\|_{L(\mathrm{X})} \|h\Big[s, \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big]\|_{\mathrm{X}} ds \\ &\leq M\int_0^t \Big[\varphi_1(s) + \varphi_2(s)\sup_{0\leq\tau\leq s}\|\bar{f}^*(\tau)\|_{\mathrm{X}}\Big]ds \\ &\leq M\int_0^t \varphi_1(s)ds + Mr\int_0^t \varphi_2(s)ds. \end{aligned}$$

Consequently $\|(\Phi_1 f) + (\Phi_2 f^*)\|_{\mathsf{B}''} \leq r$.
Similarly, for $t \in (t_j, t_{j+1}]$, $j = 1, 2, \ldots, n$, we have the estimate

$$\|(\Phi_1 f)(t) + (\Phi_2 f^*)(t)\|_X \leq \left\| \sum_{j=1}^{n} R(t-t_j) I_j(x(t_j^-) + \bar{f}(t_j^-)) \right\|_X$$
$$+ \int_0^t \|R(t-s)\|_{L(X)} \left\| h\left(s, \bar{f}^*_{\sigma(s, \bar{f}_s + x_s)} + x_{\sigma(s, \bar{f}_s + x_s)}, \int_0^s v(s,\tau, \bar{f}^*_{\sigma(\tau, \bar{f}_\tau + x_\tau)} + x_{\sigma(\tau, \bar{f}_\tau + x_\tau)}) d\tau \right) \right\|_X ds$$
$$\leq N + nMD_0 \mathsf{K}_1^* \sup_{0 \leq \tau \leq t} \|f(\tau)\|_X + M\int_0^t \varphi_1(s) ds + M \int_0^t \varphi_2(s) \sup_{0 \leq \tau \leq s} \|f(\tau)\|_X ds$$
$$\leq N + M\int_0^t \varphi_1(s) ds + \left( M\int_0^t \varphi_2(s) ds + nMD_0 \mathsf{K}_1^* \right) \sup_{0 \leq \tau \leq s} \|f(\tau)\|_X$$
$$\leq N + M\int_0^t \varphi_1(s) ds + \left( M\int_0^t \varphi_2(s) ds + nMD_0 \mathsf{K}_1^* \right) r$$
$$< r,$$

which implies that $\|(\Phi_1 f) + (\Phi_2 f^*)\|_{\mathsf{B}''} \leq r$.

**Step 2**. $\Phi_1$ is continuous on $F_r$.

Let $\{f^m\}_{m=1}^{\infty}$ be a sequence in $F_r$ with $\lim_{m\to\infty} f^m = f \in F_r$. For $t \in (t_j, t_{j+1}]$, $j = 1, 2, \ldots, n$, we obtain

$$\|(\Phi_1 f^m)(t) - (\Phi_1 f)(t)\|_X \leq \left\| \sum_{j=1}^{n} R(t-t_j) I_j(x(t_j^-) + f^m(t_j^-)) - \sum_{j=1}^{n} R(t-t_j) I_j(x(t_j^-) + f(t_j^-)) \right\|_X$$
$$\leq \sum_{j=1}^{n} \|R(t-t_j)\|_{L(X)} \|I_j(x(t_j^-) + f^m(t_j^-)) - I_j(x(t_j^-) + f(t_j^-))\|_X.$$

Since the functions $I_j$, $j = 1, 2, \ldots, n$ are continuous, thus $\lim_{m\to\infty} \Phi_1 f^m = \Phi_1 f$ in $F_r$ which implies that $\Phi_1$ is continuous on $F_r$.

**Step 3**. $\Phi_1$ is bounded.

For $t \in (t_j, t_{j+1}]$, $j = 1, 2, \ldots, n$, from Remark 3.2 , (e4), we get

$$\|(\Phi_1 f)(t)\|_X \leq \sum_{j=1}^{n} \|R(t-t_j)\|_{L(X)} \|I_j(x(t_j^-) + f(t_j^-))\|_X$$
$$\leq N + nMD_0 H \mathsf{K}_1^* r.$$

The above inequality implies that the map $\Phi_1$ is bounded.

**Step 4**. $\Phi_1(F_r)$ is equicontinuous.

Let $u, v \in (t_1, t_2]$, $u < v$, $f \in F_r$. Then by assumption (**H10**), we have

$$\begin{aligned}
&\|(\Phi_1 f)(v) - (\Phi_1 f)(u)\|_{\mathbb{X}} \\
&\le \|R(v-t_1) - R(u-t_1)\|_{L(\mathbb{X})} \|I_1(x(t_1^-) + f(t_1^-))\|_{\mathbb{X}} \\
&\le D_1(t)\|R(v-t_1) - R(u-t_1)\|_{L(\mathbb{X})} \|x(t_1^-) + f(t_1^-)\|_{\mathbb{X}} \\
&\le D_1(t)\|R(v-t_1) - R(u-t_1)\|_{L(\mathbb{X})} \sup_{t\in J}|x(t) + f(t)| \\
&\le D_0 H\|R(v-t_1) - R(u-t_1)\|_{L(\mathbb{X})} \|x(t) + f(t)\|_{\mathcal{B}_g}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_t + \bar{f}_t\|_{\mathcal{B}_g} &\le \|x_t\|_{\mathcal{B}_g} + \|\bar{f}_t\|_{\mathcal{B}_g} \\
&\le \mathsf{K}_1(t) \sup_{0\le \tau \le t} \|x(\tau)\|_{\mathbb{X}} + \mathsf{K}_2(t)\|x_0\|_{\mathcal{B}_g} + \mathsf{K}_1(t) \sup_{0\le \tau \le t} \|f(\tau)\|_{\mathbb{X}} + \mathsf{K}_2(t)\|f_0\|_{\mathcal{B}_g} \\
&\le \mathsf{K}_1(t)\Big(\|R(t)\|_{L(\mathbb{X})}|\varphi(0)|\Big) + \mathsf{K}_2(t)\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_1(t) \sup_{0\le \tau \le t} \|f(\tau)\|_{\mathbb{X}} \qquad (11) \\
&\le \mathsf{K}_1^* MH\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_1^* r \\
&\le \mathsf{K}_1^*(MH\|\varphi\|_{\mathcal{B}_g} + r) + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g}.
\end{aligned}$$

Then from the above mentioned estimates, we obtain

$$\begin{aligned}
&\|(\Phi_1 f)(v) - (\Phi_1 f)(u)\|_{\mathbb{X}} \\
&\le D_0 H(\mathsf{K}_1^*(MH\|\varphi\|_{\mathcal{B}_g} + r) + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g})\|R(v-t_1) - R(u-t_1)\|_{L(\mathbb{X})}.
\end{aligned} \qquad (12)$$

Since $R(t)$ is compact, it is operator-norm continuous. It follows that

$$\|R(v-t_j) - R(u-t_j)\|_{L(\mathbb{X})} \to 0 \text{ as } u \to v, \quad j = 1, 2, \ldots, n. \qquad (13)$$

Therefore, we deduce that

$$\|R(v-t_1) - R(u-t_1)\|_{L(\mathbb{X})} \to 0 \text{ as } u \to v. \qquad (14)$$

As a consequence, using (12) and (14), we realize that

$$\|(\Phi_1 f)(v) - (\Phi_1 f)(u)\|_{\mathbb{X}} \to 0 \text{ as } u \to v.$$

Similarly, for $u, v \in (t_j, t_{j+1}]$, $u < v$, $j = 1, \ldots, n$, $f \in F_r$, then using (11), we have

$$\begin{aligned}
&\|(\Phi_1 f)(v) - (\Phi_1 f)(u)\|_{\mathbb{X}} \\
&\le \sum_{j=1}^{n} \|R(v-t_j) - R(u-t_j)\|_{L(\mathbb{X})} \|I_j(x(t_j^-) + f(t_j^-))\|_{\mathbb{X}}. \\
&\le \sum_{j=1}^{n} \|R(v-t_j) - R(u-t_j)\|_{L(\mathbb{X})} [D_0 H\{\mathsf{K}_1^* MH\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_2^*\|\varphi\|_{\mathcal{B}_g} + \mathsf{K}_1^* r\}].
\end{aligned}$$

So, by using (13), we see that the right hand side of the above inequality converges to zero as

$u \to v$, which implies that

$$\|(\Phi_1 f)(v) - (\Phi_1 f)(u)\|_X \to 0 \text{ as } u \to v.$$

Hence, $\Phi_1(F_r)$ is equicontinuous. Thus, combining Step 2 to Step 4 together with Arzelà -Ascoli Theorem, we conclude that the operator $\Phi_1$ is compact.

**Step 5**. $\Phi_2$ is a contraction mapping.

Let $f, f^* \in F_r$ and $t \in (t_j, t_{j+1}],\ \ j = 1, 2, \ldots, n$ from the Remark 3.1, we obtain

$$\begin{aligned}
&\|(\Phi_2 f)(t) - (\Phi_2 f^*)(t)\|_X \\
&\le \int_0^t \|R(t-s)\|_{L(X)} \Big\| h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big) \\
&-h\Big(s,\ \bar{f}^*_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}^*_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_X \\
&\le K_1^* M \int_0^t [\gamma_1(s) + a\delta_1^0 \gamma_2(s)]\, ds \|f - f^*\|_{B''}.
\end{aligned}$$

Since $K_1^* M \int_0^a [\gamma_1(s) + a\delta_1^0\gamma_2(s)]ds < 1$, which implies that $\Phi_2$ is a contraction mapping. Thus by the Krasnoselskii's fixed point Theorem, we conclude that Eq. (1) has at least one mild solution on $(-\infty, a]$. Here the proof completes. ■

The last result is in agreement with Schaefer's fixed point Theorem [21, Theorem 4.3.2].

**Theorem 3.3**. *Suppose that the hypotheses (H1)-(H3) and (H8)-(H10) hold and* $nMD_0HK_1^* < 1$. *Then Eq. (1) has at least one mild solution on* $(-\infty, a]$.

*Proof.* From Theorem 3.1, we realize that the operator $\overline{\Phi}$ is defined by

$$(\overline{\Phi} f)(t) = \begin{cases}
\int_0^t R(t-s)h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} \\
\qquad + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\
R(t-t_1)I_1(x(t_1^-) + \bar{f}(t_1^-)) + \int_0^t R(t-s)\, h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + \\
\qquad x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)ds, & t \in (t_1, t_2], \\
\vdots \\
\sum_{j=1}^n R(t-t_j)I_j(x(t_j^-) + \bar{f}(t_j^-)) + \int_0^t R(t-s)h\Big(s, \bar{f}_{\sigma(s,\bar{f}_s+x_s)} \\
\qquad + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)ds, & t \in (t_n, a].
\end{cases}$$

We will divide the proof into several steps.

**Step 1**. The map $\overline{\Phi}$ is continuous on $J$.

Let $\{f^m\}_{m=1}^{\infty}$ be a sequence in $\mathsf{B}''$ such that $\|f^m - f\|_{\mathsf{B}''} \to 0$ as $m \to \infty$ in $\mathsf{B}''$, and then $r = \sup_m \|f^m\|_{\mathsf{B}''} < \infty$ and $\|f\|_{\mathsf{B}''} < r$; for every $t \in [0, t_1]$, we have

$$\begin{aligned}
&\|(\overline{\Phi}f^m)(t) - (\overline{\Phi}f)(t)\|_X \\
&\leq \int_0^t \|R(t-s)\|_{L(X)} \Big\| h\Big(s,\ \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big) \\
&\quad - h\Big(s, \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big) \Big\|_X ds \\
&\leq M \int_0^t \Big\| h\Big(s, \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big) \\
&\quad - h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big) \Big\|_X ds.
\end{aligned}$$

By assumptions (**H8**),(**H9**) and Remark 3.2, we have the following estimate:

$$\begin{aligned}
&\Big\| h\Big(s, \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big) \\
&- h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big) \Big\|_X \\
&\leq 2[\gamma_3(s) + \gamma_5(s)\int_0^t \delta_2(s)ds] + 2cn[\gamma_4(s) + \gamma_5(s)\int_0^t \delta_3(s)ds] \\
&+ 2\mathsf{K}_1^*[\gamma_4(s) + \gamma_5(s)\int_0^t \delta_3(s)ds]r.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Big\| h\Big(s, \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big) \\
&- h\Big(s, \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big) \Big\|_X \in L^1(J, \mathbb{R}^+).
\end{aligned}$$

Since the function $h$ is continuous,

$$\begin{aligned}
&h\Big(s, \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big) \\
&\quad \to h\Big(s, \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big), \text{as} \quad m \to \infty.
\end{aligned}$$

Then using the Lebesgue dominated convergence Theorem, we obtain

$$\begin{aligned}
&\int_0^t \Big\| h\Big(s,\ \bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)} + x_{\sigma(s,\bar{f}^m_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big) \\
&\cdot\ h\Big(s,\ \bar{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\ \bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big) \Big\|_X ds \to 0 \text{ as } m \to \infty
\end{aligned} \tag{15}$$

Therefore, one can see that

$$\lim_{m\to\infty}\|(\overline{\Phi}f^m)-(\overline{\Phi}f)\|_{\mathcal{B}''}=0.$$

Thus, $\overline{\Phi}$ is continuous on $[0,t_1]$.
Similarly, for every $t\in(t_j,t_{j+1}]$, we get

$$\begin{aligned}
&\|(\overline{\Phi}f^m)(t)-(\overline{\Phi}f)(t)\|_X\\
&\le\int_0^t\|R(t-s)\|_{L(X)}\Big\|h\Big(s,\bar{f}^m_{\sigma(s,\bar{f}^m_s+x_s)}+x_{\sigma(s,\bar{f}^m_s+x_s)},\int_0^s v(s,\tau,\bar{f}^m_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)}+x_{\sigma(\tau,\bar{f}^m_\tau+x_\tau)})d\tau\Big)\\
&-h\Big(s,\bar{f}_{\sigma(s,\bar{f}_s+x_s)}+x_{\sigma(s,\bar{f}_s+x_s)},\int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)}+x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_X\\
&+\Big\|\sum_{j=1}^n R(t-t_j)I_j(x(t_j^-)+\bar{f}^m(t_j^-))-\sum_{j=1}^n R(t-t_j)I_j(x(t_j^-)+\bar{f}(t_j^-))\Big\|_X.
\end{aligned}$$

By the continuity of the functions $I_j,\quad j=1,2,\ldots,n$, we have

$$I_j(x(t_j^-)+\bar{f}^m(t_j^-))\to I_j(x(t_j^-)+\bar{f}(t_j^-)),\ \text{as}\ m\to\infty. \tag{16}$$

Hence using (15) and (16), we deduce that

$$\lim_{m\to\infty}\|(\overline{\Phi}f^m)-(\overline{\Phi}f)\|_{\mathcal{B}''}=0,$$

which implies that $\overline{\Phi}$ is continuous on $(t_j,t_{j+1}],\quad j=1,2,\ldots,n$.
Finally, $\overline{\Phi}$ is continuous on $J$.

**Step 2**. $\overline{\Phi}$ maps bounded sets into bounded sets in $\mathcal{B}''$.

To this end, we will prove that for any $r>0$, there exists a constant $\Phi^*$ such that for every $f\in F_r=\{f\in\mathcal{B}'':\ \|f\|_{\mathcal{B}''}\le r\}$, we have $\|\overline{\Phi}f\|_{\mathcal{B}''}\le\Phi^*$.
For any $f\in F_r,\ t\in[0,t_1]$ jointly with Remark 3. 2, we have

$$\begin{aligned}
&\|\overline{\Phi}f(t)\|_X\\
&\le\int_0^t\|R(t-s)\|_{L(X)}\Big\|h\Big(s,\bar{f}_{\sigma(s,\bar{f}_s+x_s)}+x_{\sigma(s,\bar{f}_s+x_s)},\int_0^s v(s,\tau,\bar{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)}+x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_X ds\\
&\le M\int_0^t[\varphi_1(s)+\varphi_2(s)r]\,ds\\
&\le M\int_0^t\varphi_1(s)ds+Mr\int_0^t\varphi_2(s)ds.
\end{aligned}$$

Similarly, for $t\in(t_j,t_{j+1}],\quad j=1,2,\ldots,n$, from Remark 3.2, we obtain

$$\|\overline{\Phi}f(t)\|_X\le N+M\int_0^t\varphi_1(s)ds+[M\int_0^t\varphi_2(s)ds+nMD_0H\mathsf{K}_1^*]\,r=\Phi^*.$$

This implies that

$$\|\overline{\Phi}f\|_{\mathcal{B}''} \leq \Phi^*, \quad t \in J.$$

**Step 3**. The operator $\overline{\Phi}$ maps $F_r$ into a precompact set in X.

By the compactness of $R(t)$, the operators $R(t-t_j)I_j(x(t_j^-)+\overline{f}(t_j^-))$, $j = 1,2,\ldots,n$ are compact. Let $0 < t \leq a$ be fixed and $\tau \in (0,t)$. For $f \in F_r$, we define the operators

$$(\overline{\Phi}^\tau f)(t) = R(\tau)\int_0^{t-\tau} R(t-s-\tau)\, h\Big(s,\, \overline{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\, \overline{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)ds$$

and

$$(\widetilde{\overline{\Phi}}^\tau f)(t) = \int_0^{t-\tau} R(t-s)\, h\Big(s,\, \overline{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\, \overline{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)ds.$$

By the compactness of the operator $R(\tau)$, the set $\{(\overline{\Phi}^\tau f)(t) : f \in F_r\}$ is relatively compact in X, for every $\tau \in (0,t)$. Moreover, by the estimation **(e3)** of Remark 3.2 and Lemma 2.6, we get for each $f \in F_r$ :

$$\begin{aligned}\|(\overline{\Phi}^\tau f)(t) - (\widetilde{\overline{\Phi}}^\tau f)(t)\|_{\mathrm{X}} &\leq \int_0^{t-\tau} \|R(\tau)R(t-s-\tau) - R(t-s)\|_{L(\mathrm{X})} \\ &\times \Big\| h\Big(s,\, \overline{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\, \overline{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_{\mathrm{X}} ds \\ &\leq \varpi\tau\int_0^{t-\tau}(\varphi_1(s)+\varphi_2(s)r)ds \underset{\tau\to 0}{\longrightarrow} 0.\end{aligned}$$

So the set $\{(\widetilde{\overline{\Phi}}^\tau f)(t) : f \in F_r\}$ is precompact in X by using the total boundedness.

Applying this idea again, we obtain:

$$\begin{aligned}&\|(\overline{\Phi}f)(t) - (\widetilde{\overline{\Phi}}^\tau f)(t)\|_{\mathrm{X}} \\ &\leq \int_{t-\tau}^{t} \|R(t-s)\|_{L(\mathrm{X})} \Big\| h\Big(s,\, \overline{f}_{\sigma(s,\bar{f}_s+x_s)} + x_{\sigma(s,\bar{f}_s+x_s)}, \int_0^s v(s,\tau,\, \overline{f}_{\sigma(\tau,\bar{f}_\tau+x_\tau)} + x_{\sigma(\tau,\bar{f}_\tau+x_\tau)})d\tau\Big)\Big\|_{\mathrm{X}} ds \\ &\leq M\int_{t-\tau}^{t}(\varphi_1(s)+\varphi_2(s)r)ds \underset{\tau\to 0}{\longrightarrow} 0.\end{aligned}$$

and there are precompact sets arbitrarily close to the set $\{(\overline{\Phi}f)(t) : f \in F_r\}$. Thus, the set $\{(\overline{\Phi}f)(t) : f \in F_r\}$ is precompact in X using the Arzelà-Ascoli Theorem.

**Step 4**. $\overline{\Phi}(F_r)$ is equicontinuous.

Let $u,\ v \in [0\ ,\ t_1]$, $u < v$, from Remark 3.2, **(e3)**, we get:

$$\begin{aligned}\|\overline{\Phi}f(v)-\overline{\Phi}f(u)\|_X &\le \int_0^u \|R(v-s)-R(u-s)\|_{L(X)} \\ &\quad\times \left\| h\left(s,\overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_X ds \\ &\quad+\int_u^v \|R(v-s)\|_{L(X)} \\ &\quad\times \left\| h\left(s,\overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_X ds \\ &\le P_1+P_2,\end{aligned}$$

where

$$\begin{aligned}P_1 &= \int_0^u \|R(v-s)-R(u-s)\|_{L(X)} \\ &\quad\times \left\| h\left(s,\ \overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\ \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_X ds, \\ P_2 &= \int_u^v \|R(v-s)\|_{L(X)} \left\| h\left(s,\ \overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\ \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_X ds.\end{aligned}$$

By the dominated convergence theorem and the continuity of $R(t)$ in the operator-norm topology, we have

$$\begin{aligned}P_1 &= \int_0^u \|R(v-s)-R(u-s)\|_{L(X)} \\ &\quad\times \left\| h\left(s,\ \overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\ \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_X ds \\ &\le \int_0^u \|R(v-s)-R(u-s)\|_{L(X)}(\varphi_1(s)+\varphi_2(s))ds \to 0 \text{ as } u\to v.\end{aligned}$$

On another hand, we have

$$\begin{aligned}P_2 &= \int_u^v \|R(v-s)\|_{L(X)} \left\| h\left(s,\overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_X ds \\ &\le M\int_u^v (\varphi_1(s)+\varphi_2(s))ds \to 0 \text{ as } u\to v.\end{aligned}$$

Similarly, for $u\ ,\ v\in(t_j,t_{j+1}],$ with $j=1,2,\dots,n$, we obtain

$$\|\overline{\Phi}f(v)-\overline{\Phi}f(u)\|_X \le \sum_{j=1}^{n}\|R(v-t_j)-R(u-t_j)\|_{L(X)}\|I_j(x(t_j^-)+\overline{f}(t_j^-))\|_X+P_1+P_2.$$

By the continuity of $R(t)$ in the operator-norm topology, we conclude that

$\lim_{u\to v}\|\overline{\Phi}f(v)-\overline{\Phi}f(u)\|_{\mathrm{X}}=0$. Thus, $\overline{\Phi}(F_r)$ is equicontinuous. By these steps, using the Arzelà -Ascoli theorem, we realize that the operator $\overline{\Phi}$ is compact.

**Step 5**. The set $\mathsf{M}=\{f\in \mathsf{B}'' \ : \ f=\epsilon\overline{\Phi}f \text{ for some } 0<\epsilon<1\}$ is bounded.

Let $f\in\mathsf{M}$, then $f(t)=\epsilon\overline{\Phi}f(t)$ for some $0<\epsilon<1$. From Remark 3.2, we have for every $t\in[0,t_1]$

$$\begin{aligned}\|f(t)\|_{\mathrm{X}} \leq \epsilon\int_0^t &\|R(t-s)\|_{L(\mathrm{X})}\\ &\times \left\|h\left(s,\ \overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\ \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_{\mathrm{X}}ds\\ &\leq \epsilon M\int_0^t\left\|h\left(s,\ \overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\ \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_{\mathrm{X}}ds,\end{aligned}$$

for $(t_j,t_{j+1}]$, $\ j=1,2,\dots,n$, we obtain

$$\begin{aligned}\|f(t)\|_{\mathrm{X}} \leq \epsilon\Bigg[&\sum_{j=1}^n\|R(t-t_j)\|_{L(\mathrm{X})}\|I_j(x(t_j^-)+\overline{f}(t_j^-))\|_{\mathrm{X}}+\int_0^t\|R(t-s)\|_{L(\mathrm{X})}\\ &\times\left\|h\left(s,\ \overline{f}_{\sigma(s,\overline{f}_s+x_s)}+x_{\sigma(s,\overline{f}_s+x_s)},\int_0^s v(s,\tau,\ \overline{f}_{\sigma(\tau,\overline{f}_\tau+x_\tau)}+x_{\sigma(\tau,\overline{f}_\tau+x_\tau)})d\tau\right)\right\|_{\mathrm{X}}ds\Bigg]\\ \leq \epsilon\Bigg[&N+nMD_0H\mathsf{K}_1^*\sup_{0\leq\tau\leq t}\|f(\tau)\|_{\mathrm{X}}+M\int_0^t\varphi_1(s)ds\\ &+M\int_0^t\varphi_2(s)\sup_{0\leq\tau\leq s}\|f(\tau)\|_{\mathrm{X}}ds\Bigg].\end{aligned}$$

Likewise, for all $t\in J$, we get

$$\begin{aligned}\|f(t)\|_{\mathrm{X}} &\leq \frac{N}{1-\beta}+\frac{M}{1-\beta}\int_0^t\varphi_1(s)ds+\frac{M}{1-\beta}\int_0^t\varphi_2(s)\sup_{0\leq\tau\leq s}\|f(\tau)\|_{\mathrm{X}}ds\\ &\leq \frac{N}{1-\beta}+\frac{M}{1-\beta}\int_0^t\varphi_1(s)ds+\frac{M}{1-\beta}\int_0^t\varphi_2(s)\|f(s)\|_{\mathrm{X}}ds,\end{aligned}$$

where $\beta=n\,M\,D_0\,H\,\mathsf{K}_1^*$.

Let

$$U=\frac{N}{1-\beta}+\frac{M}{1-\beta}\int_0^t\varphi_1(s)ds, \text{ and } \ V(t)=\frac{M}{1-\beta}\int_0^t\varphi_2(s)\|f(s)\|_{\mathrm{X}}ds.$$

It is clear that $V(t)$ is a non-negative continuous function on $\mathbb{R}^+$, and by the generalized Bellman inequality, we get:

$$\|f(t)\|_{X} \leq U e^{\int_0^t V(s)ds} \leq U e^{\int_0^a V(s)ds} = S_0,$$

where $S_0$ is a constant. Obviously, the set $\mathsf{M}$ is bounded on $J$. Consequently, by the Schaefer's fixed point Theorem, we see that $\overline{\Phi}$ has a fixed point on $(-\infty, a]$. The proof is thus complete.■

## 3. Illustrative Examples

In this section, we provide two examples to illustrate our theoretical results.

**Example 1**. First, we investigate the following impulsive integrodifferential equation with state-dependent of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\zeta(t,x) = \frac{\partial^2}{\partial x^2}\zeta(t,x) + \int_0^t \vartheta(t-s)\frac{\partial^2}{\partial x^2}\zeta(s,x)ds + \int_{-\infty}^t e^{2(s-t)}\frac{\zeta(s-\sigma_1(t)\sigma_2(\|\zeta(t)\|),x)}{9}ds \\
&\quad + \int_0^t \sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}\frac{\zeta(\tau-\sigma_1(s)\sigma_2(\|\zeta(s)\|),x)}{36}d\tau ds,\ x\in[0,\pi],\ t\in[0,T],\quad t\neq t_k, \\
&\zeta(t,0) = \zeta(t,\pi) = 0,\ t\geq 0, \qquad (17)\\
&\zeta(t,x) = \varphi(t,x),\ t\leq 0,\ x\in[0,\pi], \\
&\Delta\zeta(t_j)(x) = \int_{-\infty}^{t_j} y_j(s-t_j)\cos(\zeta(t_j,x))ds,\ x\in[0,\pi],\quad j=1,2,\ldots,n,
\end{aligned}$$

where $\varphi \in \mathsf{B}_g$, $y_j = \frac{e^{2t_j}}{6}$, $j = 1,2,\ldots,n$, $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $0 < t_1 < t_2 < \cdots < t_n < T$ are pre-fixed numbers. We consider $X = L^2([0,\pi])$ with the norm $|\bullet|_{L^2}$ and define the operator $A : D(A) \subset X \to X$ by $Av = \frac{\partial^2}{\partial x^2}v$ with the domain

$$D(A) := \{v \in X : v, v' \text{ are absolutely continuous}, v'' \in X,\ v(0) = v(\pi) = 0\}.$$

Then,

$$Av = -\sum_{i=1}^{\infty} i^2\langle v, e_i\rangle e_i, \quad v \in D(A),$$

where $e_i(x) = \sqrt{\frac{2}{\pi}}\sin(ix)$, $i = 1,2,\ldots$ is the orthogonal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, thus (**H1**) is true.

Let $B : D(A) \subset X \to X$ be the operator defined by

$$B(t)y = \vartheta(t)Ay, \text{ for } t \geq 0 \text{ and } y \in D(A).$$

For phase space, we choose $g(s) = e^{2s}$, $s < 0$, then $p = \int_{-\infty}^0 g(s)ds = \frac{1}{2} < \infty$ and determine

$$\|\varphi\|_{\mathsf{B}_g} = \int_{-\infty}^0 g(s)\sup_{\theta\in[s,0]}\|\varphi(\theta)\|_{L^2}ds.$$

Here, we assume that the functions $\sigma_k : [0,\infty) \to [0,\infty), k = 1,2$ are continuous.

To represent Eq. (17) in abstract form, we define $\sigma : J \times \mathcal{B}_g \to (-\infty, a]$, $I_j : X \to X$ and $h : J \times \mathcal{B}_g \times X \to X$ by

$$\begin{aligned}\sigma(t,\varphi) &= \sigma_1(t)\sigma_2(\|\zeta(0)\|),\\ I_j(u)(x) &= \int_{-\infty}^{0} y_j(s)\cos(u(t_j)(x))ds,\ x \in [0,\pi],\quad j = 1,2,\dots,n,\\ h(t,\varphi,\Im\varphi)(x) &= \int_{-\infty}^{0} e^{2s}\frac{\varphi}{9}(x)ds + \Im\varphi(x),\end{aligned}$$

where

$$\Im\varphi(x) = \int_0^t \sin(t-s)\int_{-\infty}^{0} e^{2\theta}\frac{\varphi}{36}(x)d\theta ds.$$

If we put

$$\begin{cases} z(t)(x) = \zeta(t,x) \text{ for } t \geq 0, \text{and } 0 \leq x \leq \pi,\\ \varphi(t)(x) = \varphi(t,x),\ t \leq 0,\ 0 \leq x \leq \pi,\end{cases}$$

then, Eq. (17) is rewritten in the following abstract form

$$\begin{aligned}&\frac{d}{dt}z(t) = Az(t) + \int_0^t B(t-s)z(s)ds + h\Big(t, z_{\sigma(t,z_t)}, (\Im z)(t)\Big), \text{a.e. on } J\backslash\{t_1,t_2,\cdots,t_n\},\\ &\qquad\qquad + h(t, y_{\rho(t,y_t)}),\ \ \Delta z(t_j) = I_j(z(t_j^-)),\ j = 1,\cdots,n,\\ &z(t) = \varphi(t) \in \mathcal{B}_g.\end{aligned} \tag{18}$$

Moreover, if $\vartheta$ is bounded and $C^1$-function such that $\vartheta'$ is bounded and uniformly continuous (**H2**) is satisfied and hence, by Theorem 2.5, Eq. (17) has a unique resolvent operator $(R(t))_{t\geq 0}$ on X. For any $\varphi_1, \varphi_2 \in \mathcal{B}_g$, we have

$$\begin{aligned}&\|h(t,\varphi_1,\Im\varphi_1) - h(t,\varphi_2,\Im\varphi_2)\|_X\\ &\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2s}\left\|\frac{\varphi_1}{9} - \frac{\varphi_2}{9}\right\|ds + \int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2\theta}\left\|\frac{\varphi_1}{36} - \frac{\varphi_2}{36}\right\|d\theta ds\right)^2 dx\right)^{1/2}\\ &\leq \left(\int_0^\pi\left(\frac{1}{9}\int_{-\infty}^0 e^{2s}\sup\|\varphi_1 - \varphi_2\|ds + \frac{1}{36}\int_{-\infty}^0 e^{2s}\sup\|\varphi_1 - \varphi_2\|ds\right)^2 dx\right)^{1/2}\\ &\leq \frac{\sqrt{\pi}}{9}\|\varphi_1 - \varphi_2\|_{\mathcal{B}_g} + \frac{\sqrt{\pi}}{36}\|\varphi_1 - \varphi_2\|_{\mathcal{B}_g},\end{aligned}$$

and

$$\|v(t,s,\varphi_1)-v(t,s,\varphi_2)\|_X \le \left(\int_0^\pi \left(\|\sin(t-s)\|\int_{-\infty}^0 e^{2\theta}\left\|\frac{\varphi_1}{36}-\frac{\varphi_2}{36}\right\|d\theta ds\right)^2 dx\right)^{1/2}$$
$$\le \frac{\sqrt{\pi}}{36}\|\varphi_1-\varphi_2\|_{\mathcal{B}_g} := \delta_1\|\varphi_1-\varphi_2\|_{\mathcal{B}_g},$$

where $\delta_1 = \frac{\sqrt{\pi}}{36}$.
Hence, (**H4**) and (**H5**) are satisfied.
Similarly,

$$\|I_j(z_1(t_j)) - I_j(z_2(t_j))\|_X = \left(\int_0^\pi\left(\int_{-\infty}^0 \frac{e^{2s}}{6}(\cos(z_1(t_j)(x)-\cos(z_2(t_j)(x))\right)^2 dx\right)^{1/2}$$
$$= \left(\int_0^\pi\left(\frac{1}{12}(\cos(z_1(t_j)(x)-\cos(z_2(t_j)(x))\right)^2 dx\right)^{1/2}$$
$$\le \frac{1}{12}\left(\int_0^\pi |\cos(z_1(t_j)(x)-\cos(z_2(t_j)(x)|^2 dx\right)^{1/2}$$
$$\le \frac{1}{12}\left(\int_0^\pi \sup_{x\in[0,\pi]}|z(x)|^2|\cos(z_1(x)-\cos(z_2(x)|^2 dx\right)^{1/2}$$
$$\le \frac{\sqrt{d}}{12}\|z_1-z_2\|_X$$

where $d_i = \sup_{x\in[0,\pi]}|z(x)|^2$. Therefore, (**H6**) is satisfied. In addition, it is easy to see that (**H7**) is verified. Thus, all hypotheses are fulfilled, then from Banach contraction principle, Eq. (17) has a mild solution on $(-\infty, a]$.

**Example 2**. Now, we consider the following impulsive integrodifferential equation with state-dependent of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\zeta(t,x) = \frac{\partial^2}{\partial x^2}\zeta(t,x) + \int_0^t \vartheta(t-s)\frac{\partial^2}{\partial x^2}\zeta(s,x)ds + \frac{e^{-t}\zeta(t-\sigma_1(t)\sigma_2(\|\zeta\|))}{(9+e^t)(1+\zeta(t-\sigma_1(t)\sigma_2(\|\zeta\|)))}\\
&\qquad + \frac{t^2}{2} + \int_0^t \cos(t-s)\frac{e^s\zeta(t-\sigma_1(t)\sigma_2(\|\zeta\|))}{25}ds, \quad t\in[0,1],\ t\ne \frac{1}{2}^-,\\
&\zeta(t,0) = \zeta(t,\pi) = 0,\ t\ge 0,\\
&\zeta(t,x) = \varphi(t,x),\ t\le 0,\ x\in[0,\pi],\\
&\Delta\zeta|_{t=\frac{1}{2}^-} = \frac{\zeta\left(\frac{1^-}{2},x\right)}{49+\zeta\left(\frac{1^-}{2},x\right)},\ t\in[0,1],\quad x\in[0,\pi].
\end{aligned} \tag{19}$$

In perspective of the Example 1, we set

$$
\begin{aligned}
\zeta(t)(x) &= \zeta(t,x),\\
\sigma(t,\varphi) &= t-\sigma_1(t)\sigma_2(\|\zeta(0)\|),\\
h(t,\varphi,\Im\varphi) &= \frac{e^{-t}\varphi}{(9+e^t)(1+\varphi)}+\frac{t^2}{2}+\int_0^t \cos(t-s)\frac{e^s\varphi}{25}ds, \quad t\in[0,1],\ \varphi\in \mathsf{B}_g,\\
I_j(u) &= \frac{\zeta\left(\frac{1^-}{2},\,x\right)}{49\;+\;\zeta\left(\frac{1^-}{2},\,x\right)}.
\end{aligned}
$$

Then, for each $t\in[0,1]$, $\varphi$, $\varphi^*\in \mathsf{B}_g$, we have

$$
\begin{aligned}
\|h(t,\varphi,\Im\varphi)-h(t,\varphi^*,\Im\varphi^*)\| \le{} & \left\|\frac{e^{-t}\varphi}{(9+e^t)(1+\varphi)}-\frac{e^{-t}\varphi^*}{(9+e^t)(1+\varphi^*)}\right\|\\
&+\left\|\int_0^t\cos(t-s)\frac{e^s\varphi}{25}ds-\int_0^t\cos(t-s)\frac{e^s\varphi^*}{25}ds\right\|\\
\le{} & \frac{e^{-t}}{9+e^t}\left\|\frac{\varphi}{1+\varphi}-\frac{\varphi^*}{1+\varphi^*}\right\|+\int_0^t\cos(t-s)e^s\frac{\|\varphi-\varphi^*\|}{25}ds\\
\le{} & \frac{e^{-t}\|\varphi-\varphi^*\|}{(9+e^t)(1+\varphi)(1+\varphi^*)}+\frac{\|\varphi-\varphi^*\|}{25}\\
\le{} & \frac{e^{-t}}{(9+e^t)}\|\varphi-\varphi^*\|+\frac{\|\varphi-\varphi^*\|}{25}\\
\le{} & \frac{1}{10}\|\varphi-\varphi^*\|+\frac{1}{25}\|\varphi-\varphi^*\|
\end{aligned}
$$

and

$$
\|I_j(u)-I_j(u^*)\|\le \frac{49\|u-u^*\|}{(49+u)(49+u^*)} \le \frac{1}{49}\|u-u^*\|.
$$

Therefore the assumptions (**H4**)-(**H6**) are fulfilled with $\gamma_1=\frac{1}{10}$, $\gamma_2=\frac{1}{25}$, $\delta_1^0=1$ and $D_0=\frac{1}{49}$. Since $\gamma_1=\max\{\gamma_1(t):\ t\in[0,1]\}$ and $\gamma_2=\max\{\gamma_2(t):\ t\in[0,1]\}$. Moreover, we suppose that $n=1$, $M=1$, $\mathsf{K}_1^*=1$ and $a=1$. Thus

$$
M\,\mathsf{K}_1^*\,a\,[\gamma_1+a\,\delta_1^0\,\gamma_2]+nMD_0\approx 0.113<1.
$$

Consequently the condition (**H7**) is satisfied. Hence by the Theorem 3.1, we conclude that Eq. (19) has a unique mild solution on $[0,1]$.

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