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BSDEs With Two RCLL Reflecting Barriers Driven by a Lévy Process

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Abstract. *In this paper, we study a backward stochastic differential equation driven by a Lévy process with two right continuous and left limited reflecting barriers. We show the existence and uniqueness of solution by means of the penalization method when the coefficient is stochastic Lipschitzian. As an application, we give a fair price for the American game option.*

Key words: Doubly Reflected BSDEs, Lévy Process, Stochastic Lipschitzian Coefficient, Right Continuous With Left Limited (RCLL) Barrier.

AMS Subject Classifications: 60G20, 60H05, 60H15

1. Introduction

Backward stochastic differential equations (BSDEs in short) have been introduced in the linear case by Bismut [1] when studying adjoint equations associated with the stochastic maximum principle in optimal control. The non-linear case was developed in [2] by Pardoux and Peng . Since then the interest in BSDEs has remarkably grown for the reason that BSDEs provide a useful framework for formulating and studying many problems as in mathematical finance [3, 4], stochastic controls and stochastic games [9, 10, 11] and in partial differential equations [5, 6, 7, 8].

Furthermore, other settings of BSDEs have also been introduced. In particular, El Karoui et al. [12] have introduced the notion of reflected BSDE, which is a backward equation but the solution is forced to stay above a lower continuous stochastic process called barrier (or obstacle). Later on, Cvitanic and Karatzas [13] studied BSDEs with two reflecting barriers (DRBSDEs). In detail, a solution of such equations is a quadruple of processes (Y, Z, K^{+}, K^{-}) satisfying that

$$L_t \leq Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s \leq U_t,$$

for all $t \leq T$, where the role of the additional increasing processes K^+ and K^- is to keep the process Y between L and U in such a way that

$$\int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (U_t - Y_t)dK_t^- = 0.$$

In their seminal paper [13], the authors proved the existence and uniqueness of the solution if, on one hand, the coefficient is Lipschitz continuous and, on the other hand, either barriers are regular or satisfying the so-called Mokobodski's condition. The last condition, roughly speaking, means that there exists a difference of two non-negative supermartingales between the lower barrier L and the upper barrier U; and the regularity condition implies that the barriers can be uniformly approximated by Itô's processes. The first assumption is somehow restrictive. As for the second one, it is quite difficult to check for it in practice. For this reason, Hamadène and Hassani [14] have removed the Mokobodski's condition when they showed that if the barriers are continuous and completely separated, i.e. $L < U$, then the DRBSDEs admit a unique solution. Some of further efforts on solving DRBSDEs can be found in [15, 16, 17, 18,19, 20, 21, 22] and so on. Later on, the case of discontinuous barriers has also been studied by Hamadène et al. [23], where they actually show the existence of a solution when the obstacles and their left limits are completely separated. In addition, Essaky et al. [24] and Hamadène and Hassani [25] studied DRBSDEs driven by a Brownian motion and by an independent Poisson process. Furtherore, Hamadène and Wang [26] investigated DRBSDEs when, on one hand, the filtration is generated by Brownian motion with an independent Poisson random measure and, on the other hand, the reflecting barriers are right continuous with left limited (rcll in short) processes whose jumps are arbitrary, in the sense of being either predictable or inaccessible. Therefore the Y-component of the solution has also both types of jumps. Moreover, the increasing processes K^+ and K^- are also rcll and include the purely jumping parts $K^{+,d}$ and $K^{-,d}$ with

$$\Delta K_t^{+,d} = (Y_t - L_{t-})^- \mathbb{1}_{\{Y_{t-}=L_{t-}\}} \quad \text{and} \quad \Delta K_t^{-,d} = (Y_t - U_{t-})^+ \mathbb{1}_{\{Y_{t-}=U_{t-}\}}.$$

As to the history and latest developments, the reader is referred to [27, 28, 29, 30, 31, 32, 33, 34, 35] and to other references therein.

The theory of BSDEs driven by the Teugels martingales associated with Lévy processes has been intensively studied since Nualart and Schoutens proved in [36] the martingale representation theorem associated with Lévy processes. They also derive in [37] an existence and uniqueness result for BSDEs driven by the Teugels martingales associated with Lévy processes. Since then, many other results have been derived. We refer the reader to [38, 39, 40, 41] for BSDEs, [42, 43, 44, 45, 46] for RBSDEs and [47, 48, 49] for DRBSDEs. Note that all those studies were important both from a pure mathematical point of view and in the world of finance. They could be used for the purpose of option pricing in a Lévy market, and related with partial differential integral equations (PDIEs in short) to provide an analogue to the famous Black and Scholes formula. Moreover, an assumption of Lipschitz coefficient is made. Recently, El Jamali and El Otmani [50] provided a predictable representation property associated with Lévy process in the non-homogeneous case by leaning on the work of Nualart and Schoutens [36], and the authors established the existence and uniqueness of solution for the associated BSDE with the so-called stochastic Lipschitzian coefficient. To the best of our knowledge, Lü [51] is the first who treated RBSDEs driven by a Brownian motion and the martingales of Teugels associated with an independent Lévy process having a stochastic Lipschitzian coefficient when the barrier is continuous. Furthermore, Hu and Ren [52] considered RBSDEs related to the sub-differential operator of a lower semi-continuous convex function, driven by Teugels martingales associated with a Lévy process with stochastic Lipschitzian coefficient. Very recently, El Jamali and El Otmani [53] established the existence and uniqueness of RBSDEs with rcll barrier driven by the martingales of Teugels associated with Lévy process under a stochastic Lipschitzian coefficient.

In the present paper, our aim is to generalize the result established in [53] to DRBSDEs with two rcll barriers driven by the martingales of Teugels associated with Lévy process (DRBSDELs in short) having a stochastic Lipschitzian coefficient. We prove the existence and uniqueness of the solution by means of the penalization method and Picard's iteration. Our motivation is directed towards providing a valuation problem of an American game option in a Lévy market when the interest rate is modelled by a positive process not necessarily bounded.

The paper is organized as follows: In Section 3, we give the comparison theorem for the solutions of DRBSDELs. Section 4 is devoted to a proof of the existence and uniqueness of the solution to DRBSDELs when the coefficient is stochastic Lipschitzian. An application to computing American game option prices is discussed in Section 5.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a completed probability space on which a real-valued Lévy process $(X_t)_{t\in[0,T]}$ with càdlàg paths is defined. Let $\mathsf{F} = (\mathcal{F}_t)_{t\geq 0}$ be the right-continuous filtration generated by X: The $(\mathcal{F}_t = \sigma\{X_s; s \leq t\})$ and assume that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.
The process X is characterized by its so-called local characteristics in the Lévy-Khintchine formula. So that

$$\mathsf{E}e^{iuX_t} = e^{-t\Psi(u)} \text{ with } \Psi(u) = -ibu + \frac{\sigma^2}{2}u^2 - \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathsf{I}_{\{|x|\leq 1\}})\nu(dx).$$

Thus X is characterized by its Lévy triplet (b, σ, ν) where $b \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure defined in $\mathbb{R} \setminus \{0\}$ which satisfies
(i) $\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < +\infty$,
(ii) There exist $\varepsilon > 0$ and $\lambda > 0$ such that $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}\nu(dx) < +\infty$.
This implies that the random variables X_t have moments of all orders, i.e.

$$\int_{-\infty}^{+\infty} |x|^i \nu(dx) < \infty, \ \forall i \geq 2.$$

For more background on Lévy processes, we refer the reader to [54, 55].
We denote by $X_{t-} = \lim_{s\nearrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$, and define the power jumps of the Lévy process X by

$$X_t^{(1)} = X_t \quad \text{and} \quad X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i, \ i \geq 2.$$

Let $m_1 = \mathsf{E}[X_1] = b + \int_{|x|\geq 1} x\nu(dx)$ and $m_i = \int_{-\infty}^{+\infty} x^i \nu(dx)$ for $i \geq 2$. Let us put $Y_t^{(i)} = X_t^{(i)} - m_i t,\ i \geq 1$, the so–called Teugels martingales. We associate with the Lévy process $(X_t)_{0\leq t\leq T}$ the family of processes $(H^{(i)})_{i\geq 1}$ defined by $H_t^{(i)} = \sum_{j=1}^{i} \alpha_{ij} Y_t^{(j)}$. The coefficients α_{ij} correspond to the orthonormalization of the polynomials $1,\ x,\ x^2$ etc. with respect to the measure $\pi(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$. Specifically, the polynomials q_n defined by $q_n(x) = \sum_{k=1}^{n} \alpha_{nk} x^{k-1}$ are orthonormal with respect to the measure π:

$$\int_{\mathsf{R}} q_n(x) q_m(x) \pi(dx) = 0 \ \text{ if } \ n \neq m \quad \text{and} \quad \int_{\mathsf{R}} q_n(x)^2 \pi(dx) = 1.$$

We set

$$p_n(x) = x q_{n-1}(x) = \alpha_{n,n} x^n + \alpha_{n,n-1} x^{n-1} + \ldots + \alpha_{n,1} x.$$

The martingales $H^{(i)}$, called the orthonormalized ith-power-jump processes, are strongly orthogonal and its predictable quadratic variation process is

$$\langle H^{(i)}, H^{(j)} \rangle_t = \delta_{ij} t.$$

Let $\beta > 0$ and let $(a_t)_{t\leq T}$ be a non-negative $\mathcal{F}_t$–adapted process. Let $(A_t)_{t\leq T}$ be the increasing continuous process defined by $A_t = \int_0^t a_s^2 ds,\ \ \forall\ \ t \leq T$. Let us introduce the following appropriate spaces:

• $\mathcal{L}^2(\beta, A, \mathbb{R})$: the subspace of the $\mathcal{F}_T$–measurable and $\mathbb{R}$–valued random variable ξ such that

$$\|\xi\|^2_{\mathsf{L}^2_\beta} := \mathsf{E} e^{\beta A_T} |\xi|^2 < +\infty.$$

• $\mathsf{S}^2(\beta, A, \mathbb{R})$: the subspace of the $\mathcal{F}_t$–adapted and $\mathbb{R}$–valued processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathsf{S}^2_\beta} := \mathsf{E} \sup_{0\leq t\leq T} e^{\beta A_t} |Y_t|^2 < +\infty.$$

• $\mathsf{S}^{2,A}(\beta, A, \mathbb{R})$: the subspace of the $\mathcal{F}_t$–adapted and $\mathbb{R}$–valued processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathsf{S}^{2,A}_\beta} := \mathsf{E} \int_0^T e^{\beta A_t} |Y_t|^2 dA_t < +\infty.$$

• $\mathsf{H}^2(\beta, A, \mathbb{R})$: the space of the $\mathcal{F}_t$–adapted and $\mathbb{R}$–valued processes $(G_t)_{t\leq T}$ such that

$$\|G\|^2_{\mathsf{H}^2_\beta} := \mathsf{E} \int_0^T e^{\beta A_t} |G_t|^2 dt < +\infty.$$

• $\ell^2 = \left\{ z = (z_k)_{k\geq 1};\ \|z\|_{\ell^2} = \left(\sum_{k=1}^{\infty} |z_k|^2 \right)^{1/2} < +\infty \right\}$.

• $\mathsf{H}^2(\beta, A, \ell^2)$: the space of the $\mathcal{F}_t$–predictable and ℓ^2–valued processes $(Z_t)_{t\leq T}$ such that

$$\|Z\|^2_{\mathsf{H}^2_{\beta,\ell^2}} := \mathsf{E} \int_0^T e^{\beta A_t} \|Z_t\|^2_{\ell^2} dt = \sum_{k=1}^{\infty} \mathsf{E} \int_0^T e^{\beta A_t} |Z_t^{(k)}|^2 dt < +\infty.$$

• K^2: the subspace of the $\mathcal{F}_t$–predictable, rcll and non-decreasing processes $(K_t)_{t\leq T}$ such that

$$K_0 = 0 \quad \text{and} \quad \mathsf{E}|K_T|^2 < +\infty.$$

• $\mathsf{B}^2(\beta,A,\mathbb{R}) := \mathsf{S}^2(\beta,A,\mathbb{R}) \cap \mathsf{S}^{2,A}(\beta,A,\mathbb{R})$ is a Banach space with the norm

$$\|Y\|^2_{\mathsf{B}^2_\beta} = \|Y\|^2_{\mathsf{S}^2_\beta} + \|Y\|^2_{\mathsf{S}^{2,A}_\beta}.$$

• $\mathsf{M}^2(\beta,A,\mathbb{R}) := \mathsf{S}^{2,A}(\beta,A,\mathbb{R}) \times \mathsf{H}^2(\beta,A,\ell^2)$ is a Banach space with the norm

$$\|(Y,Z)\|^2_{\mathsf{M}^2_\beta} = \|Y\|^2_{\mathsf{S}^{2,A}_\beta} + \|Z\|^2_{\mathsf{H}^2_{\beta,\ell^2}}.$$

• $\mathsf{M}^{2,c}(\beta,A,\mathbb{R}) := \mathsf{B}^2(\beta,A,\mathbb{R}) \times \mathsf{H}^2(\beta,A,\ell^2)$ the subspace of $\mathsf{M}^2(\beta,A,\mathbb{R})$ which is Banach space equipped with the norm

$$\|(Y,Z)\|^2_{\mathsf{M}^{2,c}_\beta} = \|Y\|^2_{\mathsf{B}^2_\beta} + \|Z\|^2_{\mathsf{H}^2_{\beta,\ell^2}}.$$

In this paper, we will devote ourselves to the following doubly reflected BSDEL:

$$\left.\begin{array}{l}
Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k)} dH_s^{(k)},\\
L_t \leq Y_t \leq U_t \ \mathsf{P}\text{–a.s. for all } \ t \leq T,\\
\text{the Skorokhod condition:}\\
(i) \text{ the process } K^{\pm,c} \text{ is the continuous part of } K^{\pm} \text{ with}\\
\int_0^T (Y_t - L_t)dK_t^{+,c} = \int_0^T (U_t - Y_t)dK_t^{-,c} = 0 \ \mathsf{P}\text{–a.s.},\\
(ii) \text{ if } K^{\pm,d} \text{ is the continuous part of } K^{\pm} \text{then } K^{\pm,d} \text{ is predictable with}\\
K_t^{+,d} = \sum_{0<s<t} (Y_s - L_{s-})^- \mathsf{I}_{\{\Delta L_s < 0\}} \text{ and } K_t^{-,d} = \sum_{0<s<t} (Y_s - U_{s-})^+ \mathsf{I}_{\{\Delta L_s > 0\}},
\end{array}\right\} \quad (1)$$

where the data (ξ, f, L, U) satisfies the following:

(A. 1) The terminal value ξ is in $\mathcal{L}^2(\beta,A,\mathbb{R})$;

(A. 2) The coefficient $f : \Omega \times [0,T] \times \mathbb{R} \times \ell^2 \to \mathbb{R}$ is such that the following conditions hold:

1. For all $(y,z) \in \mathbb{R} \times \ell^2$, the function $f(.,.,y,z)$ is $\mathcal{F}_t$–progressively measurable and

$$\mathsf{E}\int_0^T e^{\beta A_s} \left| \frac{f(s,0,0)}{a_s} \right|^2 ds < +\infty.$$

2. There exist two non-negative $\mathcal{F}_t$–adapted processes $(p_t)_{t\leq T}$ and $(q_t)_{t\leq T}$ such that
   (i) for all $t \in [0,T], y, y' \in \mathbb{R}$ and $z, z' \in \ell^2$

$$|f(t,y,z) - f(t,y',z')| \leq p_t|y - y'| + q_t\|z - z'\|_{\ell^2};$$

   (ii) there exists $\epsilon > 0$ such that $a_t^2 = p_t + q_t^2 \geq \epsilon, \ \forall\, t \in [0,T]$.

(A. 3) The barriers $U := (U_t)_{t\leq T}$ and $L := (L_t)_{t\leq T}$ are $\mathcal{F}_t$–progressively measurable and rcll processes which satisfying

1. $L_T \leq \xi \leq U_T$ P–a.s.;

2. $\mathsf{E}\left[\sup_{0\le t\le T} e^{2\beta A_t}(|L_t^+|^2+|U_t^-|^2)\right]<+\infty$;
3. $L_t < U_t$ and $L_{t-} < U_{t-}$ P –a.s. for all $t \le T$.

(A. 4) There exists a semimartingale

$$R_t = R_0 + \sum_{k=1}^{\infty}\int_0^t \varphi_s^{(k)}\, dH_s^{(k)} - J_t^+ + J_t^-, \quad R_T = \xi, \tag{2}$$

with $\mathsf{E}\int_0^T \|\varphi_t\|_{\ell^2}^2 dt < +\infty$ and $J^{\pm}$ ($J_0^{\pm}=0$) are two nondecreasing continuous processes satisfying $\mathsf{E}|J_T^{\pm}|^2 < +\infty$ such that

$$L_t \le R_t \le U_t \ \mathsf{P}-\text{a.s.}, \qquad 0 \le t \le T. \tag{3}$$

Definition 2.1. Let $\beta > 0$ and $(a_t)_{t\le T}$ be a nonnegative $\mathcal{F}_t$–adapted process. A solution to DRBSDEL associated with data (ξ, f, L, U), is a quadruple of processes (Y, Z, K^+, K^-) satisfying (1) such that $(Y, Z, K^+, K^-) \in \mathsf{B}^2(\beta, A, \mathbb{R}) \times \mathsf{H}^2(\beta, A, \ell^2) \times \mathsf{K}^2 \times \mathsf{K}^2$.

Remark 2.1. The Skorokhod condition is equivalent to

$$\int_0^T (Y_{t-} - L_{t-})dK_t^+ = \int_0^T (U_{t-} - Y_{t-})dK_t^- = 0. \tag{4}$$

Indeed,

$$\begin{aligned}\int_0^T (Y_{t-} - L_{t-})dK_t^+ &= \int_0^T (Y_t - L_t)dK_t^{+,c} + \sum_{0<t\le T}(Y_{t-} - L_{t-})\Delta K_t^{+,d}\\ &= \int_0^T (Y_t - L_t)dK_t^{+,c} + \sum_{0<t\le T}(Y_{t-} - L_{t-})(Y_t - L_{t-})^- \mathsf{I}_{\{Y_{t-}=L_{t-}\}}\\ &= 0.\end{aligned}$$

In the same way, we can prove that $\int_0^T (U_{t-} - Y_{t-})dK_t^- = 0$. Conversely, we suppose that (4) holds. But

$$0 = \int_0^T (Y_{t-} - L_{t-})dK_t^+ = \int_0^T (Y_t - L_t)dK_t^{+,c} + \int_0^T (Y_{t-} - L_{t-})dK_t^{+,d}.$$

Then $\int_0^T (Y_t - L_t)dK_t^{+,c} = 0$ and

$$\begin{aligned}\Delta K_t^{+,d} = (Y_t - L_{t-})^- \mathsf{I}_{\{Y_{t-}=L_{t-}\}} &= (Y_t - L_{t-})^- \mathsf{I}_{\{Y_{t-}=L_{t-}\}\cap\{\Delta L_t<0\}}\\ &= (Y_t - L_{t-})^- \mathsf{I}_{\{\Delta L_t<0\}}.\end{aligned}$$

Similarly, we can prove that $\int_0^T (U_t - Y_t)dK_t^{-,c} = 0$ and $\Delta K_t^{-,d} = (Y_t - U_{t-})^+ \mathsf{I}_{\{\Delta U_t>0\}}$.

We point out that by $C > 0$ we always denote a finite constant whose value may change from one line to the next.

3. Comparison Theorem

In general, we do not have a comparison result for solutions of BSDEs driven by a Lévy process, reflected or not (see e.g. [56] for a counter-example). However in some specific cases, when the coefficients have some features and especially when they do not depend on z or linear on z with some assumptions, we actually have comparison. So, assume that there exists another quadruple of processes (Y',Z',K'^{+},K'^{-}) solution of the DRBSDEL (1) associated with data (ξ',f',U,L).

Theorem 3.1. *Assume that:*

1. *f is independent of z;*
2. *P-a.s. for any $t \leq T$ one has $f(t,Y'_t) \leq f'(t,Y'_t,Z'_t)$ and $\xi \leq \xi'$.*

Then $Y_t \leq Y'_t$ P-a.s. for all $t \leq T$.

Proof. Let $\bar{\Re} = \Re - \Re'$ for $\Re \in \{Y,Z,K^+,K^-,\xi\}$. From the Meyer-Itô formula (see Theorem 66 page 210 in [57]), there exists a adapted nondecreasing process $(\mathsf{A}_t)_{t\leq T}$ such that

$$|\bar{Y}_t^+|^2 = |\bar{Y}_0^+|^2 - 2\int_0^t \bar{Y}_s^+[f(s,Y_s) - f'(s,Y'_s,Z'_s)]ds - 2\int_0^t \bar{Y}_{s-}^+ d\bar{K}_s^+$$
$$+ 2\int_0^t \bar{Y}_{s-}^+ d\bar{K}_s^- + 2\sum_{k=1}^{\infty}\int_0^t \bar{Y}_{s-}^+ \bar{Z}_s^{(k)} dH_s^{(k)} + \mathsf{A}_t.$$

The integration by parts formula implies that

$$e^{\beta A_t}|\bar{Y}_t^+|^2 = e^{\beta A_T}|\xi^+|^2 - \beta\int_t^T e^{\beta A_s}|\bar{Y}_s^+|^2 dA_s + 2\int_t^T e^{\beta A_s}\bar{Y}_s^+[f(s,Y_s) - f'(s,Y'_s,Z'_s)]ds$$
$$+ 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}^+ d\bar{K}_s^+ - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}^+ d\bar{K}_s^- - 2\sum_{k=1}^{\infty}\int_t^T e^{\beta A_s}\bar{Y}_{s-}^+ \bar{Z}_s^{(k)} dH_s^{(k)}$$
$$-\int_t^T e^{\beta A_s} d\mathsf{A}_s.$$

Taking the expectation on both sides above, we obtain

$$\mathsf{E}e^{\beta A_t}|\bar{Y}_t^+|^2 + \beta\mathsf{E}\int_t^T e^{\beta A_s}|\bar{Y}_s^+|^2 dA_s + \mathsf{E}\int_t^T e^{\beta A_s} d\mathsf{A}_s$$
$$= 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_s^+[f(s,Y_s) - f(s,Y'_s)]ds + 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_s^+[f(s,Y'_s) - f'(s,Y'_s,Z'_s)]ds$$
$$+ 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_{s-}^+([dK_s^+ + dK'^{-}_s] - [dK_s^- + dK'^{+}_s]).$$

Remark that if $Y > Y'$ then $Y > L$ and $U > Y'$ which implies that $dK^{+,c} = 0$ and $dK'^{-,c} = 0$. Also, when the purely discontinuous $K^{+,d}$ increases at s we should have $Y_{s-} = L_{s-}$ and when $K'^{-,d}$ increases at s we should have $Y'_{s-} = U_{s-}$, which implies that

$$\sum_{0<s<t} \bar{Y}^+_{s-}\left[\Delta K^{+,d}_s + \Delta K'^{-,d}_s\right]$$
$$= \sum_{0<s<t}(L_{s-} - Y'_{s-})^+ \Delta K^{+,d}_s + \sum_{0<s<t}(Y_{s-} - U_{s-})^+ \Delta K'^{-,d}_s = 0.$$

Moreover, by using the assumption $(\mathsf{A}.2)(2)$, we get

$$\mathsf{E}e^{\beta A_t}|\bar{Y}^+_t|^2 + (\beta - 2)\mathsf{E}\int_t^T e^{\beta A_s}|\bar{Y}^+_s|^2 dA_s + \mathsf{E}\int_t^T e^{\beta A_s} d\mathsf{A}_s \leq 0.$$

Then, for $\beta \geq 2$ we obtain $\mathsf{E}[e^{\beta A_t}|\bar{Y}^+_t|^2] \leq 0$, and thus $Y_t \leq Y'_t$ P-a.s. for all $t \leq T$. ■

Remark 3.1. BSDE and Reflected BSDE driven by a Lévy process is a particular case of (1). Therefore we can deduce, from Theorem 2.1, a comparison result to:

BSDEL	$L = -\infty$ and $U = +\infty$	$K^+ = 0$ and $K^- = 0$
RBSDEL with one lower barrier	$U = +\infty$	$K^- = 0$
RBSDEL with one upper barrier	$L = -\infty$	$K^+ = 0$

If f is Lipschitzian, we have also the comparison result for BSDEL (Hamadène and Zhao [44]), RBSDEL (Zhou [49]) and DRBSDEL (Ren and El Otmani [48]).

4. Existence & Uniqueness of the Solution

In this section, we are going to establish the existence and uniqueness of solution to the DRBSDELs (1) associated with parameters (ξ, f, L, U).

4.1. Uniqueness of the solution

The aim of this part is to prove the uniqueness of the solutions to DRBSDELs. In doing so, let (Y, Z, K^+, K^-) and (Y', Z', K'^+, K'^-) be two solutions of the DRBSDEL (1) with data (ξ, f, L, U) and (ξ', f', L', U') respectively. First, we give an a priori estimate of the difference of the above solutions which will be useful in the sequel.

Proposition 4.1. *Under the condition* $(\mathsf{A}.2)(2)$*, there exists a positive constant C such that for each* $\beta > 2$

$$\mathsf{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t}|\bar{Y}_t|^2\right] + \mathsf{E}\int_0^T e^{\beta A_s}|\bar{Y}_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|\bar{Z}_s\|^2_{\ell^2} ds$$
$$\leq C\left\{\mathsf{E}e^{\beta A_T}|\bar{\xi}|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{f(s, Y'_s, Z'_s) - f'(s, Y'_s, Z'_s)}{a_s}\right|^2 ds\right.$$
$$\left. + \mathsf{E}\int_0^T e^{\beta A_s}\bar{L}_{s-} d\bar{K}^+_s - \mathsf{E}\int_0^T e^{\beta A_s}\bar{U}_{s-} d\bar{K}^-_s\right\}$$

where $\bar{\Re} = \Re - \Re'$*, for* $\Re \in \{Y, Z, K^+, K^-, L, U, \xi\}$.

Proof. Applying Itô's formula to $e^{\beta A_t}|\bar{Y}_t|^2$, taking the expectation and using condition $(\mathsf{A}.2)(2)$, we obtain for some $\varrho > 0$ and $\rho > 0$

$$\begin{aligned}
&\mathsf{E}e^{\beta A_t}|\bar{Y}_t|^2 + \beta\mathsf{E}\int_t^T e^{\beta A_s}|\bar{Y}_s|^2 dA_s + \mathsf{E}\int_t^T e^{\beta A_s}\|\bar{Z}_s\|^2_{\ell^2} ds \\
&\leq \mathsf{E}e^{\beta A_T}|\bar{\xi}|^2 + 2\mathsf{E}\int_t^T e^{\beta A_s}|\bar{Y}_s|(p_s|\bar{Y}_s| + q_s\|\bar{Z}_s\|_{\ell^2})ds \\
&\quad + 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_s(f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s))ds \\
&\quad + 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{K}^+_s - 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{K}^-_s \\
&\leq \mathsf{E}e^{\beta A_T}|\bar{\xi}|^2 + (2+\varrho)\mathsf{E}\int_t^T e^{\beta A_s}|\bar{Y}_s|^2 dA_s + \frac{1}{\rho}\mathsf{E}\int_t^T e^{\beta A_s}\|\bar{Z}_s\|^2_{\ell^2} ds \\
&\quad + \frac{1}{\varrho}\mathsf{E}\int_t^T e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s)}{a_s}\right|^2 ds \\
&\quad + 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{K}^+_s - 2\mathsf{E}\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{K}^-_s
\end{aligned}$$

where $1 < \rho < 2$. By using the facts that

$$\bar{Y}_{s-}d\bar{K}^+_s = (L_{s-} - Y'_{s-})dK^+_s - (Y_{s-} - L'_{s-})dK'^+_s \leq (L_{s-} - L'_{s-})d\bar{K}^+_s$$

and $\bar{Y}_{s-}d\bar{K}^-_s \geq (U_{s-} - U'_{s-})d\bar{K}^-_s$, and by choosing $\beta > 0$ such that $\beta > \varrho + 2$ we get

$$\begin{aligned}
&\sup_{0\leq t\leq T}\mathsf{E}e^{\beta A_t}|\bar{Y}_t|^2 + \mathsf{E}\int_0^T e^{\beta A_s}|\bar{Y}_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|\bar{Z}_s\|^2_{\ell^2} ds \\
&\leq C\Bigg\{\mathsf{E}e^{\beta A_T}|\bar{\xi}|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,Y'_s,Z'_s) - f'(s,Y'_s,Z'_s)}{a_s}\right|^2 ds \\
&\qquad + \mathsf{E}\int_0^T e^{\beta A_s}\bar{L}_{s-}d\bar{K}^+_s - \mathsf{E}\int_0^T e^{\beta A_s}\bar{U}_{s-}d\bar{K}^-_s\Bigg\}.
\end{aligned}$$

Finally, the result follows from the Burkholder-Davis-Gundy inequality. ■

Corollary 4.1. *Under the assumption* $(\mathsf{A}.2)(2)$*, if* (Y,Z,K^+,K^-) *verifies DRBSDEL (1), then we have*

$$\begin{aligned}
&\mathsf{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}|Y_t|^2\right] + \mathsf{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_s\|^2_{\ell^2} ds + \mathsf{E}|K_T|^2 \\
&\leq C\left\{\mathsf{E}e^{\beta A_T}|\xi|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds + \mathsf{E}\sup_{0\leq t\leq T} e^{2\beta A_t}(|L^+_t|^2 + |U^-_t|^2)\right\},
\end{aligned}$$

where $K = K^+ - K^-$.

Proof. Note that $(Y',Z',K'^+,K'^-) = (0,0,0,0)$ is a solution of the DRBSDEL (1) with data

$(\xi', f', L', U') = (0,0,0,0)$. Then by Proposition 4.1, we can write for some $\eta > 0$

$$\mathsf{E} \sup_{0\le t\le T} \left[e^{\beta A_t}|Y_t|^2\right] + \mathsf{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_s\|^2_{\ell^2} ds$$

$$\le C\left\{\mathsf{E}e^{\beta A_T}|\xi|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds + \mathsf{E}\int_0^T e^{\beta A_s} L_{s-} dK_s^+ \right.$$

$$\left. - \mathsf{E}\int_0^T e^{\beta A_s} U_{s-} dK_s^- \right\}$$

$$\le C\left\{\mathsf{E}e^{\beta A_T}|\xi|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds + \eta\, \mathsf{E}\sup_{0\le t\le T} e^{2\beta A_t}\left(|L_t^+|^2 + |U_t^-|^2\right)\right.$$

$$\left. + \frac{1}{\eta}\left(\mathsf{E}|K_T^+|^2 + \mathsf{E}|K_T^-|^2\right)\right\}. \tag{5}$$

On the other hand, from the equation

$$K_T = K_T^+ - K_T^- = Y_0 - \xi - \int_0^T f(s, Y_s, Z_s) ds + \sum_{k=1}^{\infty}\int_0^T Z_s^{(k)} dH_s^{(k)},$$

we have

$$\mathsf{E}|K_T|^2 \le 4\left\{\mathsf{E}\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2 + \mathsf{E}e^{\beta A_T}|\xi|^2 + \frac{3}{\beta}\mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds\right.$$

$$\left. + \frac{3}{\beta}\mathsf{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \left(1 + \frac{3}{\beta}\right)\mathsf{E}\int_0^T e^{\beta A_s}\|Z_s\|^2_{\ell^2} ds\right\}. \tag{6}$$

Finally, the desired result is obtained by combining the estimates (5) and (6) for $\eta > 4C\left(1 + \frac{3}{\beta}\right)$. ■

Proposition 4.2. *Assume that* $(\mathsf{A}.2)(2)$ *and* $(\mathsf{A}.3)(3)$ *hold. Then, there exists at most one quadruple* (Y, Z, K^+, K^-) *solution of the DRBSDEL (1).*

Proof. Let (Y, Z, K^+, K^-) and (Y', Z', K'^+, K'^-) be two solutions of the DRBSDEL (1) associated with data (ξ, f, L, U). From Proposition 4.1, we have immediately $Y = Y'$ and $Z = Z'$. Next, we have also $K^+ = K'^+$ and $K^- = K'^-$; Indeed, by using Remark 4.1, we can write for all $t \le T$

$$\int_0^t (U_{s-} - L_{s-}) dK_s^- = \int_0^t (Y_{s-} - L_{s-}) dK_s^- = -\int_0^t (Y_{s-} - L_{s-}) d(K_s^+ - K_s^-)$$

$$= -\int_0^t (Y_{s-} - L_{s-}) d(K_s'^+ - K_s'^-) = \int_0^t (Y'_{s-} - L_{s-}) dK_s'^- = \int_0^t (U_{s-} - L_{s-}) dK_s'^-.$$

Since $L < U$, we conclude that $K^- = K'^-$. Similarly, we can prove that $K^+ = K'^+$. Thus we have uniqueness of the solution. ■

4. 2. **Uniqueness of the solution via a penalization method**

The aim here is to prove the existence of a solution to DRBSDEL (1) associated with parameters (ξ, f, L, U) via the penalization method and the fixed point theorem where the barriers are completely separated. We first consider the special case when the coefficient does not depend on (y,z), that is $f(t,y,z) = g(t)$.

Theorem 4. 1. *Assume that* $\frac{g}{a} \in H^2(\beta, A, \mathbb{R})$, (A.1), (A.3) *and* (A.4) *hold. Then the DRBSDEL associated with parameters* (ξ, g, L, U) *admits a unique solution.*

Proof. We consider for any $m,n \geq 1$ the following generalized BSDEL which is a penalized version of (1):

$$Y_t^{m,n} = \xi + \int_t^T g(s)ds + m\int_t^T (Y_s^{m,n} - L_s)^- ds - n\int_t^T (Y_s^{m,n} - U_s)^+ ds$$
$$-\sum_{k=1}^{\infty} \int_t^T Z_s^{(k)m,n} dH_s^{(k)}. \tag{7}$$

Set $g_{m,n}(s,y) = g(s) + m(y - L_s)^- - n(y - U_s)^+$. Obviously that $g_{m,n}$ is $(m+n)$-Lipschitz on y, consequently, by Theorem A.1 (see Appendix A), there exists a unique process $(Y^{m,n}, Z^{m,n}) \in \mathsf{M}^{2,c}(\beta, A, \mathbb{R})$ solution of BSDEL (7).

We denote $K_t^{+,m,n} = m\int_0^t (Y_s^{m,n} - L_s)^- ds$ and $K_t^{-,m,n} = n\int_0^t (Y_s^{m,n} - U_s)^+ ds$.

Step 1: A priori estimates on the sequence $(Y^{m,n}, Z^{m,n}, K^{+,m,n}, K^{-,m,n})$:
There exists a constant $C > 0$ such that

$$\sup_{m,n\geq 1} \left\{ \| Y^{m,n} \|^2_{\mathsf{B}^2_\beta} + \| Z^{m,n} \|^2_{\mathsf{H}^2_{\beta,\ell^2}} + \mathsf{E}|K^{+,m,n}|^2 + \mathsf{E}|K^{-,m,n}|^2 \right\}$$
$$\leq C\left(\|\xi\|^2_{\mathsf{L}^2_\beta} + \left\| \frac{g}{a} \right\|^2_{\mathsf{H}^2_\beta} + \|L^+\|^2_{\mathsf{S}^2_{2\beta}} + \|U^-\|^2_{\mathsf{S}^2_{2\beta}} + \|\varphi\|^2_{\mathsf{H}^2_{\ell^2}} + \mathsf{E}|J_T^+|^2 + \mathsf{E}|J_T^-|^2 \right). \tag{8}$$

First, we consider the following RBSDEL associated with (ξ, g, L) for all $t \leq T$

$$\begin{cases} \overline{Y}_t = \xi + \int_t^T g(s)ds + \overline{K}_T - \overline{K}_t - \sum_{k=1}^{\infty} \int_t^T \overline{Z}_s^{(k)} dH_s^{(k)} \\ \overline{Y}_t \geq L_t, \quad \int_0^T (\overline{Y}_t - L_t) d\overline{K}_t^c = 0 \quad \text{and} \quad \overline{K}_t^d = \sum_{0<s<t} (\overline{Y}_s - L_{s-})^- \mathsf{I}_{\{\Delta L_s < 0\}}. \end{cases} \tag{9}$$

From Theorem B.1 (see Appendix B), there exists a unique triplet of processes $(\overline{Y}, \overline{Z}, \overline{K}) \in \mathsf{B}^2(\beta, A, \mathbb{R}) \times \mathsf{H}^2(\beta, A, \ell^2) \times \mathsf{K}^2$ solution of RBSDEL (9). We consider the penalization equation associated with the RBSDEL (9), for any $m \in \mathsf{N}$

$$\overline{Y}_t^m = \xi + \int_t^T g(s)ds + m\int_t^T (\overline{Y}_s^m - L_s)^- ds - \sum_{k=1}^{\infty} \int_t^T \overline{Z}_s^{(k),m} dH_s^{(k)}.$$

The comparison Theorem A.2 (see Appendix A) implies that $\overline{Y}_t^0 \leq \overline{Y}_t^m \leq \overline{Y}_t^{m+1}$ and $Y_t^{m,n} \leq \overline{Y}_t^m$ for all $t \leq T$. Therefore, $\overline{Y}_t^m \nearrow \overline{Y}_t$ for all $t \leq T$ as $m \to +\infty$. Hence $Y_t^{m,n} \leq \overline{Y}_t$. With same way, from Corollary B.1 (see Appendix B), there exists a unique triplet of processes $(\underline{Y}, \underline{Z}, \underline{K}) \in \mathsf{B}^2(\beta, A, \mathbb{R}) \times \mathsf{H}^2(\beta, A, \ell^2) \times \mathsf{K}^2$ solution to the following RBSDEL with data (ξ, g, U) for all $t \leq T$

$$\begin{cases} \underline{Y}_t = \xi + \int_t^T g(s)ds - (\underline{K}_T - \underline{K}_t) - \sum_{k=1}^{\infty} \int_t^T \underline{Z}_s^{(k)} dH_s^{(k)} \\ \underline{Y}_t \le U_t, \quad \int_0^T (U_t - \underline{Y}_t) d\underline{K}_t^c = 0 \quad \text{and} \quad \underline{K}_t^d = \sum_{0<s<t} (\underline{Y}_s - U_{s-})^+ \mathbb{1}_{\{\Delta U_s > 0\}}. \end{cases} \tag{10}$$

We consider the penalization equation associated with the RBSDEL (10), for any $n \in \mathbb{N}$

$$\underline{Y}_t^n = \xi + \int_t^T g(s)ds - n\int_t^T (\underline{Y}_s^n - U_s)^+ ds - \sum_{k=1}^{\infty} \int_t^T \underline{Z}_s^{(k),n} dH_s^{(k)}.$$

From the comparison Theorem A.2 (see Appendix A), we have $Y_t^{m,n} \ge \underline{Y}_t$ for all $t \le T$. Consequently, by using the estimates (B.2) and (B.3) (see Appendix B), we derive that

$$\begin{aligned} \mathbb{E} \sup_{0 \le t \le T} e^{\beta A_t} |Y_t^{m,n}|^2 &\le \max \left\{ \mathbb{E} \sup_{0 \le t \le T} e^{\beta A_t} |\overline{Y}_t|^2, \mathbb{E} \sup_{0 \le t \le T} e^{\beta A_t} |\underline{Y}_t|^2 \right\} \\ &\le C \left(2\mathbb{E} e^{\beta A_T} |\xi|^2 + 2\mathbb{E} \int_0^T e^{\beta A_s} \left| \frac{g(s)}{a_s} \right|^2 ds + \mathbb{E} \sup_{0 \le t \le T} e^{2\beta A_t} \{ |L_t^+|^2 + |U_t^-|^2 \} \right). \end{aligned} \tag{11}$$

On the other hand, Itô's formula implies that

$$\begin{aligned} & e^{\beta A_t} |Y_t^{m,n}|^2 + \beta \int_t^T e^{\beta A_s} |Y_s^{m,n}|^2 dA_s + \int_t^T e^{\beta A_s} \| Z_s^{m,n} \|_{\ell^2}^2 ds \\ &= e^{\beta A_T} |\xi|^2 + 2\int_t^T e^{\beta A_s} Y_s^{m,n} g(s) ds + 2m \int_t^T e^{\beta A_s} Y_{s-}^{m,n} (Y_s^{m,n} - L_s)^- ds \\ & -2n \int_t^T e^{\beta A_s} Y_{s-}^{m,n} (Y_s^{m,n} - U_s)^+ ds - 2\sum_{k=1}^{\infty} \int_t^T e^{\beta A_s} Y_{s-}^{m,n} Z_s^{(k),m,n} dH_s^{(k)} \\ & - \sum_{k,k'=1}^{\infty} \int_t^T e^{\beta A_s} Z_s^{(k),m,n} Z_s^{(k'),m,n} d[H^{(k)}, H^{(k')}]_s. \end{aligned} \tag{12}$$

Using the basic inequality $2ab \le \alpha a^2 + \frac{b^2}{\alpha}$, and taking expectation on both sides of (12), we obtain

$$\begin{aligned} & \frac{\beta}{2} \mathbb{E} \int_0^T e^{\beta A_s} |Y_s^{m,n}|^2 dA_s + \mathbb{E} \int_0^T e^{\beta A_s} \| Z_s^{m,n} \|_{\ell^2}^2 ds \\ & \le \mathbb{E} e^{\beta A_T} |\xi|^2 + \frac{2}{\beta} \mathbb{E} \int_0^T e^{\beta A_s} \left| \frac{g(s)}{a_s} \right|^2 ds + \alpha \mathbb{E} \sup_{0 \le t \le T} e^{2\beta A_t} (|L_t^+|^2 + |U_t^-|^2) \\ & \quad + \frac{1}{\alpha} \mathbb{E} \left| \int_0^T m(Y_s^{m,n} - L_s)^- ds \right|^2 + \frac{1}{\alpha} \mathbb{E} \left| \int_0^T n(Y_s^{m,n} - U_s)^+ ds \right|^2. \end{aligned} \tag{13}$$

Now, we need to estimate $\mathbb{E}\left[\int_0^T m(Y_s^{m,n} - L_s)^- ds \right]^2 + \mathbb{E}\left[\int_0^T n(Y_s^{m,n} - U_s)^+ ds \right]^2$. For this, let us introduce the following stopping times

$$\begin{cases} \tau_0 = 0, \\ \tau_{2i+1} = \inf\{ t > \tau_{2i} \text{ such that } Y_t^{m,n} = L_t \} \wedge T, \quad i \ge 0 \\ \tau_{2i+2} = \inf\{ t > \tau_{2i+1} \text{ such that } Y_t^{m,n} = U_t \} \wedge T, \quad i \ge 0, \end{cases}$$

with $\mathbb{P}(Y^{m,n} < L) = \mathbb{P}(Y^{m,n} > U) = 0$. Clearly that $(\tau_i)_{i \ge 0}$ is strictly nondecreasing on the set

$G = \bigcap_{i\geq 0}\{\tau_i < T\}$ since L and U are completely separated barriers. In addition, $(\tau_i)_{i\geq 0}$ is of stationary type (i.e. $\forall\omega \in \Omega$, there exists $i_0(\omega)$ such that $\forall i \geq i_0(\omega)$, $\tau_i(\omega) = \tau_{i+1}(\omega) = T$). Indeed, we show that $\mathsf{P}(G) = 0$. Assuming that $\mathsf{P}(G) > 0$; therefore, for $\omega \in G$, there exists two sequences of real numbers $(t_i(\omega))_{i\geq 0}$ and $(t_i^{'}(\omega))_{i\geq 0}$ belongs to $[\tau_{i-1}, \tau_i]$ such that $Y^{m,n}_{t_i} = U_{t_i} \wedge U_{t_i-} = U_{t_i} - (\Delta U_{t_i})^+$ and $Y^{m,n}_{t_i^{'}} = L_{t_i^{'}} \vee L_{t_i^{'}-} = L_{t_i^{'}} + (\Delta L_{t_i^{'}})^-$. Now as $(t_i)_{i\geq 0}$ and $(t_i^{'})_{i\geq 0}$ are not of stationary type since $(\tau_i)_{i\geq 0}$ is nondecreasing sequence, then taking the limit as $i \to +\infty$ to obtain that $Y^{m,n}_{\tau-}(\omega) = L_{\tau-}(\omega) = U_{\tau-}(\omega) = Y^{m,n}_{\tau-}(\omega)$. Then $L_{\tau-}(\omega) = U_{\tau-}(\omega)$, but this is contradiction since P-a.s. $\forall t \leq T, L_{t-} < U_{t-}$. We deduce that $\mathsf{P}(G) = 0$.

Next, let $p \geq 1$ be real number, and let $\nu^p_{\tau_{2i}}$ and $\nu^p_{\tau_{2i+1}}$ be a stopping times defined by :
$\nu^p_{\tau_{2i}} = \inf\left\{t > \tau_{2i} \;\mid\; Y^{m,n}_t \leq L_t + \frac{1}{p}\right\} \wedge T,$
$\nu^p_{\tau_{2i+1}} = \inf\left\{t > \tau_{2i+1} \;\mid\; Y^{m,n}_t \geq U_t - \frac{1}{p}\right\} \wedge T.$
Then for all $t \in [\tau_{2i}, \nu^p_{\tau_{2i}}[$, we have $Y^{m,n}_{t-} > L_{t-}$ and for all $t \in [\tau_{2i+1}, \nu^p_{\tau_{2i+1}}[$, we have $Y^{m,n}_{t-} < U_{t-}$. Consequently, it holds true that $Y^{m,n} > L$ on $[\tau_{2i}, \tau_{2i+1} \wedge \nu^p_{\tau_{2i}}[$ and $Y^{m,n} < U$ on $[\tau_{2i+1}, \tau_{2i+2} \wedge \nu^p_{\tau_{2i+1}}[$. Moreover, remark that the sequence of stopping times $\nu^p_{\tau_{2i}}$ and $\nu^p_{\tau_{2i+1}}$ converge to τ_{2i+1} and τ_{2i+2} respectively as $p \to +\infty$. It follows the BSDEL (7) that

$$Y^{m,n}_{\tau_{2i}} = Y^{m,n}_{\tau_{2i+1}} + \int_{\tau_{2i}}^{\tau_{2i+1}} g(s)ds - n\int_{\tau_{2i}}^{\tau_{2i+1}} (Y^{m,n}_s - U_s)^+ds - \sum_{k=1}^{\infty}\int_{\tau_{2i}}^{\tau_{2i+1}} Z^{(k)m,n}_s dH^{(k)}_s. \tag{14}$$

On the other hand, by (3), we have the following

$Y^{m,n}_{\tau_{2i}} \geq R_{\tau_{2i}}$	$\{\tau_{2i} < T\}$
$Y^{m,n}_{\tau_{2i}} = R_{\tau_{2i}} = \xi$	$\{\tau_{2i} = T\}$
$Y^{m,n}_{\tau_{2i+1}} \leq R_{\tau_{2i+1}}$	$\{\tau_{2i+1} < T\}$
$Y^{m,n}_{\tau_{2i+1}} = R_{\tau_{2i+1}} = \xi$	$\{\tau_{2i+1} = T\}$

.

Combining (14) with (2), we get

$$n\int_{\tau_{2i}}^{\tau_{2i+1}} (Y^{m,n}_s - U_s)^+ds \leq R_{\tau_{2i+1}} - R_{\tau_{2i}} + \int_{\tau_{2i}}^{\tau_{2i+1}}|g(s)|ds - \sum_{k=1}^{\infty}\int_{\tau_{2i}}^{\tau_{2i+1}} Z^{(k)m,n}_s dH^{(k)}_s$$
$$\leq \int_{\tau_{2i}}^{\tau_{2i+1}}|g(s)|ds + \sum_{k=1}^{\infty}\int_{\tau_{2i}}^{\tau_{2i+1}}\left\{\varphi^{(k)}_s - Z^{(k)m,n}_s\right\}dH^{(k)}_s + J^-_{\tau_{2i+1}} - J^-_{\tau_{2i}}.$$

Using the fact that $Y^{m,n} < U$ on $[\tau_{2i+1}, \tau_{2i+2}[$ and summing in i, we obtain

$$\mathsf{E}\Big[n\int_0^T(Y^{m,n}_s - U_s)^+ds\Big]^2 \leq 4\bigg(\frac{1}{\beta}\mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{g(s)}{a_s}\right|^2 ds + \mathsf{E}\int_0^T\|\varphi_s\|^2_{\ell^2}ds$$
$$+ \mathsf{E}\int_0^T e^{\beta A_s}\|Z^{m,n}_s\|^2_{\ell^2}ds + \mathsf{E}|J^-_T|^2\bigg). \tag{15}$$

Similarly, we can obtain

$$\mathsf{E}\Big[m\int_0^T(Y^{m,n}_s - L_s)^-ds\Big]^2 \leq 4\bigg(\frac{1}{\beta}\mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{g(s)}{a_s}\right|^2 ds + \mathsf{E}\int_0^T\|\varphi_s\|^2_{\ell^2}ds$$
$$+ \mathsf{E}\int_0^T e^{\beta A_s}\|Z^{m,n}_s\|^2_{\ell^2}ds + \mathsf{E}|J^+_T|^2\bigg). \tag{16}$$

The desired result can be obtained by combining the estimates (15), (16) with (13) for $\alpha > 8$, and adding the estimate (11).

Step 2: Due to the convergence results in El Jamali and El Otmani [53], we have

$$\left.\begin{array}{l} Y^{m,n} \to Y^n \text{ in } \mathsf{B}^2(\beta,A,\mathbb{R}) \text{ as } m \text{ tends to } +\infty, \\ Z^{m,n} \to Z^n \text{ in } \mathsf{H}^2(\beta,A,\ell^2) \text{ as } m \text{ tends to } +\infty \\ K^{+,m,n} \to K^{+,n} \text{ in } \mathsf{K}^2 \text{ as } m \text{ tends to } +\infty. \end{array}\right\} \tag{17}$$

Furthermore, the process $(Y^n,Z^n,K^{+,n})$ is the unique solution of the RBSDEL associated with parameters (ξ,g_n,L) with $g_n(s,y) = g(s) - n(y-U_s)^+$, that is for all $t \leq T$

$$\begin{cases} Y^n_t = \xi + \int_t^T g_n(s,Y^n_s)ds + K^{+,n}_T - K^{+,n}_t - \sum_{k=1}^{\infty}\int_t^T Z^{(k),n}_s dH^{(k)}_s \\ Y^n_t \geq L_t, \quad \int_0^T (Y^n_t - L_t)dK^{+,n,c}_t = 0 \quad \text{and} \quad K^{+,n,d}_t = \sum_{0<s<t}(Y^n_s - L_{s-})^- \mathsf{I}_{\{\Delta L_s<0\}}. \end{cases}$$

Let $K^{-,n}_t = n\int_0^t (Y^n_s - U_s)^+ds$. Thanks to (17) and (8), there exists a constant $C > 0$ such that

$$\begin{aligned} \sup_{n\geq 1} & \left\{ \|Y^n\|^2_{\mathsf{B}^2_\beta} + \|Z^n\|^2_{\mathsf{H}^2_{\beta,\ell^2}} + \mathsf{E}|K^{+,n}|^2 + \mathsf{E}|K^{-,n}|^2 \right\} \\ & \leq C\left(\|\xi\|^2_{\mathsf{L}^2_\beta} + \left\|\frac{g}{a}\right\|^2_{\mathsf{H}^2_\beta} + \|L^+\|^2_{\mathsf{S}^2_{2\beta}} + \|U^-\|^2_{\mathsf{S}^2_{2\beta}} + \|\varphi\|^2_{\mathsf{H}^2_{\ell^2}} + \mathsf{E}|J^+_T|^2 + \mathsf{E}|J^-_T|^2 \right). \end{aligned} \tag{18}$$

Step 3: We must to prove that:

$$\mathsf{E}\left[\sup_{0\leq t\leq T} e^{2\beta A_t}|(Y^n_t - U_t)^+|^2\right] \xrightarrow[n\to+\infty]{} 0. \tag{19}$$

Let $(\widehat{Y}^n,\widehat{Z}^n,\widehat{K}^{+,n})$ be the unique solution of the following RBSDEL

$$\begin{cases} \widehat{Y}^n_t = \xi + \int_t^T \left(g(s) + n(U_s - \widehat{Y}^n_s)\right)ds + \widehat{K}^{+,n}_T - \widehat{K}^{+,n}_t - \sum_{k=1}^{\infty}\int_t^T \widehat{Z}^{(k),n}_s dH^{(k)}_s \\ \widehat{Y}^n_t \geq L_t, \quad \int_0^T (\widehat{Y}^n_t - L_t)d\widehat{K}^{+,n,c}_t = 0 \quad \text{and} \quad \widehat{K}^{+,n,d}_t = \sum_{0<s<t}(\widehat{Y}^n_s - L_{s-})^- \mathsf{I}_{\{\Delta L_s<0\}}, \end{cases}$$

and let $(\widehat{Y}^{m,n},\widehat{Z}^{m,n})$ be solution of its penalized BSDEL. Since $(U_s - \widehat{Y}^n_s) = (U_s - \widehat{Y}^n_s)^+ - (\widehat{Y}^n_s - U_s)^+$, then, from comparison Theorem 6 (see Appendix A) we have $Y^{m,n}_t \leq \widehat{Y}^{m,n}_t$ P-a.s. for all $t \leq T$. As m tends to $+\infty$, we get $Y^n_t \leq \widehat{Y}^n_t$ P-a.s. for all $t \leq T$. Next, from Itô's formula we have

$$e^{-nt}\widehat{Y}^n_t = e^{-nT}\xi + \int_t^T e^{-ns}(g(s) + nU_s)ds + \int_t^T e^{-ns}d\widehat{K}^{+,n}_s - \sum_{k=1}^{\infty}\int_t^T e^{-ns}\widehat{Z}^{(k),n}_s dH^{(k)}_s$$

So, we can write (see Lemma 4.2 in [43])

$$\begin{aligned} \widehat{Y}^n_t = \operatorname*{ess}_{\tau\in\mathsf{T}_{t,T}} \mathsf{E}\Big[& e^{-n(T-t)}\xi\mathsf{I}_{\{\tau=T\}} + e^{-n(\tau-t)}L_\tau\mathsf{I}_{\{\tau<T\}} \\ & + \int_t^\tau e^{-n(s-t)}g(s)ds + n\int_t^\tau e^{-n(s-t)}U_s ds/\mathcal{F}_t \Big] \end{aligned}$$

where $\mathsf{T}_{t,T}$ is the set of stopping times with values in $[t,T]$. Recall that $L_t \leq R_t \leq U_t$, then for each stopping time $v \leq T$ we have

$$\widehat{Y}_{\nu}^{n} \leq \mathsf{E}\Big[\int_{\nu}^{T} e^{-n(s-\nu)}|g(s)|ds + n\int_{\nu}^{T} e^{-n(s-\nu)}(U_s - R_s)ds/\mathcal{F}_{\nu}\Big] + \operatorname*{ess\,sup}_{\tau\in\mathsf{T}_{\nu,T}} \mathsf{E}\Big[e^{-n(T-\nu)}\xi \mathsf{I}_{\{\tau=T\}} + e^{-n(\tau-\nu)}R_{\tau}\mathsf{I}_{\{\tau<T\}} + n\int_{\nu}^{\tau} e^{-n(s-\nu)}R_s ds/\mathcal{F}_{\nu}\Big].$$

Since

$$e^{-n(\tau-\nu)}R_{\tau} + n\int_{\nu}^{\tau} e^{-n(s-\nu)}R_s ds = R_{\nu} + \int_{\nu}^{\tau} e^{-n(s-\nu)}dR_s,$$

then we have

$$\widehat{Y}_{\nu}^{n} \leq \mathsf{E}\Big[\int_{\nu}^{T} e^{-n(s-\nu)}|g(s)|ds + n\int_{\nu}^{T} e^{-n(s-\nu)}(U_s - R_s)ds/\mathcal{F}_{\nu}\Big] + e^{-n(T-\nu)}\xi \mathsf{I}_{\{\nu=T\}} + R_{\nu}\mathsf{I}_{\{\nu<T\}} + \operatorname*{ess\,sup}_{\tau\in\mathsf{T}_{\nu,T}} \mathsf{E}\Big[\int_{\nu}^{\tau} e^{-n(s-\nu)}d(J_s^{+} + J_s^{-})/\mathcal{F}_{\nu}\Big].$$

It is easily seen that

$$n\int_{\nu}^{T} e^{-n(s-\nu)}(U_s - R_s)ds \xrightarrow[n\to+\infty]{} (U_{\nu} - R_{\nu})\,\mathsf{I}_{\{\nu<T\}}, \qquad \mathsf{P}-a.s. \text{ and in } \mathcal{L}^2.$$

Moreover, the conditional expectation converges also in L^2. In addition, by Hölder inequality, we have

$$\Big|\int_{\nu}^{T} e^{-n(s-\nu)}|g(s)|ds\Big|^2 \leq \Big(\int_{\nu}^{T} e^{\beta A_s}\Big|\frac{g(s)}{a_s}\Big|^2 ds\Big)\Big(\int_{\nu}^{T} e^{-2n(s-\nu)-\beta A_s}dA_s\Big).$$

Thus $\mathsf{E}\Big[\int_{\nu}^{T} e^{-n(s-\nu)}|g(s)|ds|\mathcal{F}_{\nu}\Big] \xrightarrow[n\to\infty]{} 0$, P-a.s. Also, we have

$$\mathsf{E}\Big[\int_{\nu}^{\tau} e^{-n(s-\nu)}d(J_s^{+} + J_s^{-})/\mathcal{F}_{\nu}\Big] \xrightarrow[n\to+\infty]{} 0 \qquad \text{in } \mathcal{L}^2.$$

We conclude that

$$Y_{\nu} \leq \widehat{Y}_{\nu} = \lim_{n\to+\infty} \widehat{Y}_{\nu}^{n} \leq \xi\,\mathsf{I}_{\{\nu=T\}} + U_{\nu}\,\mathsf{I}_{\{\nu<T\}} = U_{\nu}, \qquad \mathsf{P}-a.s.$$

From this and the section theorem of Dellacherie-Meyer ([58], p. 220, Theorem 86), we deduce that $Y_t \leq U_t$, P-a.s. for all $t \leq T$. Hence $(Y_t^n - U_t)^+$ decreases to 0. From generalized Dini's lemma (see page 202 in [59]), we have $\sup_{0\leq t\leq T} e^{\beta A_t}(Y_t^n - U_t)^+ \searrow 0$, P-a.s. for all $t \leq T$. Since $|(Y_t^n - U_t)^+| \leq |Y_t^0| + |U_t^+|$, the result follows from the dominated convergence theorem.

Step 4: The sequence $(Y^n, Z^n, K^{+,n}, K^{-,n})_{n\geq 0}$ converge to some process $(Y, Z, K^+, K^-) \in \mathsf{B}^2(\beta, A, \mathbb{R}) \times \mathsf{H}^2(\beta, A, \ell^2) \times \mathsf{K}^2 \times \mathsf{K}^2$.

For $n, p \geq 0$, applying Itô's formula to $e^{\beta A_t}|Y_t^n - Y_t^p|^2$ and taking the expectation, we obtain

$$\mathsf{E}e^{\beta A_t}|Y_t^n - Y_t^p|^2 + \beta\mathsf{E}\int_t^T e^{\beta A_s}|Y_s^n - Y_s^p|^2 dA_s + \mathsf{E}\int_t^T e^{\beta A_s}\|Z_s^n - Z_s^p\|_{\ell^2}^2 ds$$
$$\leq 2\mathsf{E}\int_t^T e^{\beta A_s}(Y_{s-}^n - Y_{s-}^p)(p(Y_s^p - U_s)^+ - n(Y_s^n - U_s)^+)ds$$

$$+ 2\mathsf{E}\int_t^T e^{\beta A_s}(Y^n_{s-} - Y^p_{s-})(dK^{+,n}_s - dK^{+,p}_s)$$
$$\leq 2\mathsf{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}(Y^n_t - U_t)^+ K^{-,p}_T\right] + 2\mathsf{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}(Y^p_t - U_t)^+ K^{-,n}_T\right]$$
$$+ 2\mathsf{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}(Y^n_t - L_t)^- K^{+,p}_T\right] + 2\mathsf{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}(Y^p_t - L_t)^- K^{+,n}_T\right].$$

Next, by virtue of (19) and take into consideration (18), we have

$$\mathsf{E}\int_t^T e^{\beta A_s}|Y^n_s - Y^p_s|^2 dA_s + \mathsf{E}\int_t^T e^{\beta A_s}\|Z^n_s - Z^p_s\|^2_{\ell^2} ds \xrightarrow[n,p\to+\infty]{} 0,$$

which implies that $(Y^n, Z^n)_{n\geq 0}$ is a Cauchy sequence in $\mathsf{S}^{2,A}(\beta, A, \mathbb{R}) \times \mathsf{H}^2(\beta, A, \ell^2)$. So, there exists $(Y, Z) \in \mathsf{S}^{2,A}(\beta, A, \mathbb{R}) \times \mathsf{H}^2(\beta, A, \ell^2)$ such that

$$\mathsf{E}\int_t^T e^{\beta A_s}|Y^n_s - Y_s|^2 dA_s + \mathsf{E}\int_t^T e^{\beta A_s}\|Z^n_s - Z_s\|^2_{\ell^2} ds \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, by the Burkholder-Davis-Gundy's inequality, there exists a universal positive constant c such that

$$2\mathsf{E}\sup_{0\leq t\leq T}\left|\sum_{k=1}^{\infty}\int_0^t e^{\beta A_s}(Y^n_{s-} - Y^p_{s-})\left(Z^{(k),n}_s - Z^{(k),p}_s\right)dH^{(k)}_s\right|$$
$$\leq \frac{1}{2}\mathsf{E}\sup_{0\leq t\leq T} e^{\beta A_t}|Y^n_t - Y^p_t|^2 + 2c^2\mathsf{E}\int_0^T e^{\beta A_s}\|Z^n_s - Z^p_s\|^2_{\ell^2} ds$$

and

$$\mathsf{E}\sup_{0\leq t\leq T}\left|\sum_{k,k'=1}^{\infty}\int_0^t e^{\beta A_s}\left(Z^{(k),n}_s - Z^{(k),p}_s\right)\left(Z^{(k'),n}_s - Z^{(k'),p}_s\right)d\left[H^{(k)}, H^{(k')}\right]_s\right|$$
$$\leq c\mathsf{E}\int_0^T e^{\beta A_s}\|Z^n_s - Z^p_s\|^2_{\ell^2} ds.$$

Then, by taking the supremum over $t \in [0, T]$ and then the expectation after application of Itô formula to $e^{\beta A_t}|Y^n_t - Y^p_t|^2$, one can derive that

$$\mathsf{E}\sup_{0\leq t\leq T} e^{\beta A_t}|Y^n_t - Y^p_t|^2 \xrightarrow[n,p\to+\infty]{} 0.$$

Consequently, there exists $Y \in \mathsf{S}^2(\beta, A, \mathrm{R})$ such that $Y^n \to Y$ as $n \to +\infty$.

Now, let us demonstrate the convergence of the increasing processes $K^{+,n}$ and $K^{-,n}$. The comparison Theorem B.2 (see Appendix B) shows that $K^{+,n}_t \leq K^{+,n+1}_t$ for all $t \leq T$; therefore, there exists a process K^+ such that $K^{+,n} \nearrow K^+$, and it follows from the generalized Dini's lemma (see page 202 in [59]), that

$$\mathsf{E}\sup_{0\leq t\leq T}|K^{+,n}_t - K^+_t|^2 \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, the quadruplet $(Y^n, Z^n, K^{+,n}, K^{-,n})$ satisfies the equation

$$K_t^{-,n} = Y_t^n - Y_0^n + \int_0^t g(s)ds + K_t^{+,n} - \sum_{k=1}^{\infty}\int_0^t Z_s^{(k),n} dH_s^{(k)}.$$

Using the Burkholder-Davis-Gundy's inequality, for $n,p \geq 0$, we have

$$\begin{aligned}\mathsf{E}\sup_{0\leq t\leq T}|K_t^{-,n} - K_t^{-,p}|^2 &\leq 4\Big\{\mathsf{E}\sup_{0\leq t\leq T} e^{\beta A_t}|Y_t^n - Y_t^p|^2 + \mathsf{E}|Y_0^n - Y_0^p|^2 \\ &\quad + \mathsf{E}\sup_{0\leq t\leq T}|K_t^{+,n} - K_t^{+,p}|^2 + c\mathsf{E}\int_0^T e^{\beta A_s}\|Z_s^n - Z_s^p\|_{\ell^2}^2 ds\Big\}. \\ &\xrightarrow[n,p\to+\infty]{} 0.\end{aligned}$$

Then, there exists a processes K^- such that $K^{-,n} \to K^-$ in K^2.

Step 5: The limiting process (Y,Z,K^+,K^-) is a unique solution to the DRBSDEL (1):
In Step 4, we have proved that $(Y^n,Z^n,K^{+,n},K^{-,n}) \longrightarrow (Y,Z,K^+,K^-)$ as $n \to +\infty$ in $\mathsf{B}^2(\beta,A,\mathbb{R}) \times \mathsf{H}^2(\beta,A,\ell^2) \times \mathsf{K}^2 \times \mathsf{K}^2$. Then by passing to limits as $m,n \to +\infty$ in (7), we get

$$Y_t = \xi + \int_t^T g(s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k)} dH_s^{(k)}.$$

On the other hand, from Step 3, it holds true that $L_t \leq Y_t \leq U_t$ for all $t \leq T$. Now, we would like to show the Skorokhod's conditions. We proceed in the same way as [53] did for one-reflected BSDELs. We define the processes

$$\begin{cases}\eta_t^+ := L_t \mathsf{I}_{\{t<T\}} + \xi \mathsf{I}_{\{t=T\}} + \int_0^t g(s)ds, \\ \eta_t^- := -U_t \mathsf{I}_{\{t<T\}} + \xi \mathsf{I}_{\{t=T\}} + \int_0^t g(s)ds.\end{cases}$$

Note that $\eta^\pm$ are rcll and uniformly square integrable processes. With the notation of the Snell envelope, recall that $\mathsf{S}(\eta) := \underset{\tau\in\mathsf{T}_{0,T}}{\mathrm{ess}}\ \mathsf{E}[\eta_\tau/\mathcal{F}_t]$. Now $\mathsf{S}(\eta^\pm)$ is a supermartingale (the smallest rcll supermartingale which dominates the process $\eta^\pm$). Then, by the Doob-Meyer decomposition theorem, there exist an $\mathcal{F}_t$-adapted rcll increasing processes $K^\pm$ with $\mathsf{E}[K_T^\pm] < +\infty$ and $K_0^\pm = 0$ such that

$$\mathsf{S}(\eta_t^\pm) = \mathsf{E}\Big[\xi + \int_0^T g(s)ds + K_T^\pm/\mathcal{F}_t\Big] - K_t^\pm = M_t^\pm - K_t^{\pm,c} - K_t^{\pm,d}.$$

the filtration is generated by a Lévy process, the jumping times of $(M_t^\pm)_{t\leq T}$ are those of the power-jump processes associated with the Lévy process. Therefore, when $K^{\pm,d}$ jump, the process $\mathsf{S}(.)$ has the same jump. Then $\{\Delta K^{\pm,d} > 0\} \subset \{\mathsf{S}_-(\eta^\pm) = \eta_-^\pm\}$ and

$$\begin{aligned}\int_0^T (Y_{s-} - L_{s-})dK_s^{+,d} &= \sum_{0<s<T}(Y_{s-} - L_{s-})\mathsf{I}_{\{\Delta K_s^{+,d}>0\}}\Delta K_s^{+,d} \\ &= \sum_{0<s<T}(Y_{s-} - L_{s-})(\mathsf{S}_s(\eta^+) - \eta_{s-}^+)^- \mathsf{I}_{\{\mathsf{S}_{s-}(\eta^+)=\eta_{s-}^+\}} \\ &= 0.\end{aligned}$$

Similarly we have $\int_0^T (U_{s-} - Y_{s-})dK_s^{-,d} = 0$. On the other hand, by some property of the Snell envelope (see Lemma 5.1 in [33]), we get

$$0 = \int_0^T (\mathsf{S}(\eta_t^+) - \eta_t^+) dK_t^{+,c} = \int_0^T (Y_t - L_t) dK_t^{+,c}$$

and

$$0 = \int_0^T (\mathsf{S}(\eta_t^-) - \eta_t^-) dK_t^{-,c} = \int_0^T (U_t - Y_t) dK_t^{-,c}.$$

Theorem 4.1 is then proved. ■

The main result of this paper is what follows.

Theorem 4. 2. *Assume that* (A.1)*,* (A.2)*,* (A.3) *and* (A.4) *hold. Then the DRBSDEL (1) associated with parameters* (ξ, f, L, U) *has a unique solution.*

Proof. We consider the following sequence of DRBSDELs

$$Y_t^{n+1} = \xi_T + \int_t^T f(s, Y_s^n, Z_s^n) ds + K_T^{+,n+1} - K_t^{+,n+1} - (K_T^{-,n+1} - K_t^{-,n+1}) - \sum_{k=1}^{\infty} \int_t^T Z_s^{(k),n+1} dH_s^{(k)}, \tag{20}$$

with $(Y^0, Z^0) = (0,0)$ and under the constraints

- $L_t \leq Y_t^{n+1} \leq U_t$ P –a.s.
- $\int_0^T (Y_t^{n+1} - L_t) dK_t^{+,n+1,c} = \int_0^T (U_t - Y_t^{n+1}) dK_t^{-,n+1,c} = 0$ a.s.
- $K_t^{+,n+1,d} = \sum_{0<s<t} (Y_s^{n+1} - L_{s-})^{-} \mathbf{1}_{\{\Delta L_s < 0\}}$ a.s.
- $K_t^{-,n+1,d} = \sum_{0<s<t} (Y_s^{n+1} - U_{s-})^{+} \mathbf{1}_{\{\Delta U_s > 0\}}$ a.s.

By the recursive principe and from Theorem 4.1, the DRBSDEL (20) admit a unique solution $(Y^n, Z^n, K^{+,n}, K^{-,n})$.

Using Picard's iteration, we shall prove that (Y^n, Z^n) is a Cauchy sequence in the Banach space $\mathsf{M}^{2,c}(\beta, A, \mathbb{R})$. For each $n \geq p \geq 0$, put $\Re^{n,p} = \Re^n - \Re^p$ for $\Re \in \{Y, Z, K^+, K^-\}$. Let us prove that

$$\|(Y^{n+1,p+1}, Z^{n+1,p+1})\|^2_{\mathsf{M}^{2,c}_\beta} \leq c_\beta \|(Y^{n,p}, Z^{n,p})\|^2_{\mathsf{M}^{2,c}_\beta},$$

where $c_\beta \in]0,1[$ for a suitable choice of β. Observe that

$$Y_t^{n+1,p+1} = \int_t^T [f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)] ds + K_T^{+,n+1,p+1} - K_t^{+,n+1,p+1} - (K_T^{-,n+1,p+1} - K_t^{-,n+1,p+1}) - \sum_{k=1}^{\infty} \int_t^T Z_s^{(k),n+1,p+1} dH_s^{(k)}.$$

By applying Itô's formula we get

$$e^{\beta A_t}|Y_t^{n+1,p+1}|^2 + \beta\int_t^T e^{\beta A_s}|Y_s^{n+1,p+1}|^2 dA_s + \int_0^T e^{\beta A_s}\|Z_s^{n+1,p+1}\|_{\ell^2}^2 ds$$
$$= 2\int_t^T e^{\beta A_s} Y_s^{n+1,p+1}[f(s,Y_s^n,Z_s^n) - f(s,Y_s^p,Z_s^p)]ds$$
$$+2\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1} dK_s^{+,n+1,p+1} - 2\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1} dK_s^{-,n+1,p+1}$$

$$-2\sum_{k=1}^{\infty}\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1} Z_s^{(k),n+1,p+1} dH_s^{(k)}$$
$$-\sum_{k,k'=1}^{\infty}\int_t^T e^{\beta A_s} Z_s^{(k),n+1,p+1} Z_s^{(k'),n+1,p+1} d\Big([H^{(k)},H^{(k')}]_s - \langle H^{(k)},H^{(k')}\rangle_s\Big).$$

But it holds true that

- $$Y_s^{n+1,p+1}[f(s,Y_s^n,Z_s^n) - f(s,Y_s^p,Z_s^p)]ds$$
$$\le (\beta-1)|Y_s^{n+1,p+1}|^2 dA_s + \frac{2}{\beta-1}\int_t^T e^{\beta A_s}[a_s^2|Y_s^{n,p}|^2 + \|Z_s^{n,p}\|_{\ell^2}^2]ds;$$

- $Y_{s-}^{n+1,p+1}(dK_s^{+,n+1,p+1} - dK_s^{-,n+1,p+1}) \le 0.$

Then, on one hand,

$$\mathsf{E}\int_0^T e^{\beta A_s}|Y_S^{n+1,p+1}|^2 dA_S + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_S^{n+1,p+1}\|_{\ell^2}^2 ds$$
$$\le \frac{2}{\beta-1}\Big(\mathsf{E}\int_0^T e^{\beta A_s}|Y_S^{n,p}|^2 dA_S + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_S^{n,p}\|_{\ell^2}^2 ds\Big). \tag{21}$$

On the other hand, we have

$$e^{\beta A_t}|Y_t^{n+1,p+1}|^2 \le \frac{2}{\beta-1}\left(\int_0^T e^{\beta A_s}|Y_s^{n,p}|^2 dA_s + \int_0^T e^{\beta A_s}\|Z_s^{n,p}\|_{\ell^2}^2 ds\right)$$
$$-2\sum_{k=1}^{\infty}\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1} Z_s^{(k),n+1,p+1} dH_s^{(k)}$$
$$-\sum_{k,k'=1}^{\infty}\int_t^T e^{\beta A_s} Z_s^{(k),n+1,p+1} Z_s^{(k'),n+1,p+1} d[H^{(k)},H^{(k')}]_s.$$

But by Burkholder-Davis-Gundy's inequality there exists a universal constant c such that

$$\mathsf{E}\sup_{0\le t\le T} e^{\beta A_t}|Y_t^{n+1,p+1}|^2$$
$$\le \frac{4}{\beta-1}((2c^2+c)\vee 1)\left(\mathsf{E}\sup_{0\le t\le T} e^{\beta A_t}|Y_t^{n,p}|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_s^{n,p}\|_{\ell^2}^2 ds\right).$$

It follows that

$$\|(Y^{n+1,p+1},Z^{n+1,p+1})\|^2_{\mathsf{M}^{2,c}_\beta} \le \left(\frac{2}{\beta-1}\right)\vee\left(\frac{4}{\beta-1}((2c^2+c)\vee 1)\right)\|(Y^{n,p},Z^{n,p})\|^2_{\mathsf{M}^{2,c}_\beta}.$$

We choose $\beta > 4((2c^2+c)\vee 1)+1$ and by iterating the last inequality, we deduce that the sequence (Y^n,Z^n) is a Cauchy sequence and converges to (Y,Z).which is, with K^+ and K^-, the unique solution of the DRBSDEL (1). ■

Remark 4. 1.
- The Brownian motion $(B_t)_{t\le T}$ is characterized by $(b,\sigma,0)$, then we have also the existence and uniqueness result of doubly reflected BSDE with rcll barriers in a Brownian framework.
- Poisson process and compound Poisson process are characterized by $(0,0,\lambda\delta_1(dx))$ and $(0,0,\lambda\mathsf{P}_{\mathbf{Z}_1})$ respectively, where δ_1 is Dirac measure at point 1, and $\mathsf{P}_{\mathbf{Z}_1}$ is the distribution of the random variable $\mathbf{Z}_1$. Then we have also the existence and uniqueness result of doubly reflected BSDE with jumps and rcll barriers.

5. Application to Computing American Game Option Prices

In finance, an American option is a contract which enables its buyer (holder) to exercise it at any time up to the maturity. An American game option gives additionally the right to the option seller (writer, issuer) to cancel it early by paying for this a prescribed penalty. If the buyer exercises at time t he receives the amount $L_t \ge 0$ from the seller and if the seller exercises at time t before the buyer he must pay to the buyer the amount $U_t \ge L_t$ so that $U_t - L_t$ is viewed as a penalty imposed on the seller for cancelation of the contract. If both exercise at the expiry time T then the buyer may claim the amount ξ. American game option was first introduced by Kifer [60] and studied later by several authors (see for example [61, 62]).
In this part, we study the application to computing American game option prices by using a backward stochastic differential equation (1). For it, we consider the market with two assets: one is a non-risky asset with price S^0 and the other is a risky asset with price S defined on $[0,T]$ by

$$\begin{cases} dS^0_t = r_t S^0_t dt, \quad S^0_0 = 1; \\ dS_t = S_{t-}dX_t, \quad S_0 = x > 0, \end{cases} \tag{22}$$

where $(r_t)_{t\le T}$ is a positive process representing the interest rate. We suppose that there exists a risk-neutral measure $\mathbb{Q}$ such that :
- $\left\{\tilde{X}_t := X_t - \int_0^t r_s ds; t\in[0,T]\right\}$ is a Lévy process under $\mathbb{Q}$.
- $\tilde{S} = \frac{S}{S^0}$ is a $\mathbb{Q}$-martingale.

We enlarge the market with the processes $S^{(i)}$ defined for $i\ge 2$ as $S^{(i)}_t = e^{\int_0^t r_s ds}\tilde{H}^{(i)}_t$ with $(\tilde{H}^{(i)})^\infty_{i=1}$ is the orthonormalized power jump processes associated with the Lévy process $\tilde{X}$. Note that $\tilde{S}^{(i)} = \frac{S^{(i)}}{S^0}$ is a $\mathbb{Q}$-martingale. It follows that the stock $\tilde{S}$ and the power jump assets $(\tilde{S}^{(i)})_{i\ge 2}$ remain arbitrage-free.
Let us have a look at the pricing problem of an American game option driven by the market model which is given by (22). If the seller decides to cancel at a stopping time $\tau \le T$ and the

buyer exercises at a stopping time $\theta \leq T$ then the former pays to the latter the amount :

$$\Im(\tau,\theta) = e^{-R_{t,T}} g(\tilde{S}_T)\, \mathbb{1}_{\{\theta=\tau=T\}} + e^{-R_{t,\tau}} \mathbf{U}(\tilde{S}_\tau)\, \mathbb{1}_{\{\tau<\theta\}} + e^{-R_{t,\theta}} \mathbf{L}(\tilde{S}_\theta)\, \mathbb{1}_{\{\theta\leq\tau\}}$$

where $R_{t,s} = \int_t^s r_u du$. For all $t \leq T$, we consider the following BSDEL with two reflecting barriers

$$\begin{cases} Y_t = g(\tilde{S}_T) - \int_t^T r_s Y_s ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \sum_{k=1}^\infty \int_t^T Z_s^{(k)} d\tilde{H}_s^{(k)}. \\ \mathbf{L}(\tilde{S}_t) \leq Y_t \leq \mathbf{U}(\tilde{S}_t) \quad \mathbb{Q}-\text{a.s..} \\ \int_0^T (Y_t - \mathbf{L}(\tilde{S}_t)) dK_t^{+,c} = \int_0^T (\mathbf{U}(\tilde{S}_t) - Y_t) dK_t^{-,c} = 0. \\ K_t^{+,d} = \sum_{0<s<t} (Y_s - \mathbf{L}(\tilde{S}_{s-}))^- \mathbb{1}_{\{\Delta \mathbf{L}(\tilde{S}_s)<0\}} \\ \text{and } K_t^{-,d} = \sum_{0<s<t} (Y_s - \mathbf{U}(\tilde{S}_{s-}))^+ \mathbb{1}_{\{\Delta \mathbf{U}(\tilde{S}_s)>0\}}. \end{cases}$$

Let us assume that there exists $p \geq 1$ and $\kappa_p > 0$ such that

$$|g(x)| + |\mathbf{L}(x)| + |\mathbf{U}(x)| \leq \kappa_p(1 + |x|^p), \quad \forall\, x \in \mathbb{R}.$$

Let $\mathrm{T}_{t,T}$ is the set of all $\mathcal{F}_t$-stopping times taking values in $[t,T]$ and τ_t^*, θ_t^* be the stopping times defined as follows :

$$\tau_t^* = \inf\{s \geq t : Y_s = \mathbf{U}(\tilde{S}_s)\} \wedge T \quad \text{and} \quad \theta_t^* = \inf\{s \geq t : Y_s = \mathbf{L}(\tilde{S}_s)\} \wedge T.$$

The fair price of the American game option is given by

$$V_t = \mathrm{E}_{\mathbb{Q}}[\Im(\tau_t^*, \theta_t^*)/\mathcal{F}_t].$$

main result of this section now follows.

Theorem 5. 1. *The value of an American game option is given by*

$$Y_t = V_t = \inf_{\tau\in \mathrm{T}_{t,T}} \sup_{\theta\in \mathrm{T}_{t,T}} \mathrm{E}_{\mathbb{Q}}[\Im(\tau,\theta)/\mathrm{F}_t] = \sup_{\theta\in \mathrm{T}_{t,T}} \inf_{\tau\in \mathrm{T}_{t,T}} \mathrm{E}_{\mathbb{Q}}[\Im(\tau,\theta)/\mathcal{F}_t].$$

Proof. Using the integration by parts formula, we obtain

$$\begin{aligned} Y_t = {} & e^{-\int_t^{\tau_t^*\wedge\theta_t^*} r_u du} Y_{\tau_t^*\wedge\theta_t^*} + \int_t^{\tau_t^*\wedge\theta_t^*} e^{-\int_t^s r_u du} dK_s^+ - \int_t^{\tau_t^*\wedge\theta_t^*} e^{-\int_t^s r_u du} dK_s^- \\ & - \sum_{k=1}^\infty \int_t^{\tau_t^*\wedge\theta_t^*} e^{-\int_t^s r_u du} Z_s^{(k)} d\tilde{H}_s^{(k)}. \end{aligned}$$

But $\int_t^{\tau_t^*\wedge\theta_t^*} (Y_t - \mathbf{L}(\tilde{S}_{s-})) dK_t^{+,c} = \int_t^{\tau_t^*\wedge\theta_t^*} (\mathbf{U}(\tilde{S}_{s-}) - Y_t) dK_t^{-,c} = 0$, therefore $K_{\tau_t^*\wedge\theta_t^*}^{+,c} - K_t^{+,c} = 0$, $K_{\tau_t^*\wedge\theta_t^*}^{-,c} - K_t^{-,c} = 0$ and

- $K_{\tau_t^*\wedge\theta_t^*}^{+,d} - K_t^{+,d} = \sum_{t\leq s<\tau_t^*\wedge\theta_t^*} (Y_s - \mathbf{L}(\tilde{S}_{s-}))^- \mathbb{1}_{\{Y_{s-}=\mathbf{L}(\tilde{S}_{s-})\}} = 0$,
- $K_{\tau_t^*\wedge\theta_t^*}^{-,d} - K_t^{-,d} = \sum_{t\leq s<\tau_t^*\wedge\theta_t^*} (Y_s - \mathbf{U}(\tilde{S}_{s-}))^+ \mathbb{1}_{\{Y_{s-}=\mathbf{U}(\tilde{S}_{s-})\}} = 0$.

Indeed, $Y_{s-} \neq \mathbf{U}(\tilde{S}_{s-})$ and $Y_{s-} \neq \mathbf{L}(\tilde{S}_{s-})$ for all $s \in [t, \tau_t^* \wedge \theta_t^*[$. Then, after taking conditional expectation, we get

$$\begin{aligned} Y_t &= \mathsf{E}_{\mathbb{Q}}\left(e^{-\int_t^{\tau_t^* \wedge \theta_t^*} r_u du} Y_{\tau_t^* \wedge \theta_t^*} / \mathcal{F}_t \right) \\ &= \mathsf{E}_{\mathbb{Q}}\left(e^{-\int_t^{\tau_t^*} r_u du} \mathbf{U}(\tilde{S}_{\tau_t^*}) \mathsf{I}_{\{\tau_t^* < \theta_t^*\}} + e^{-\int_t^{\theta_t^*} r_u du} \mathbf{L}(\tilde{S}_{\theta_t^*}) \mathsf{I}_{\{\theta_t^* \leq \tau_t^*\}} \right. \\ &\qquad \left. + e^{-\int_t^T r_u du} g(\tilde{S}_T) \mathsf{I}_{\{\theta_t^* = \tau_t^* = T\}} / \mathcal{F}_t \right) = \mathsf{E}_{\mathbb{Q}}[\Im(\tau_t^*, \theta_t^*) / \mathcal{F}_t]. \end{aligned} \tag{23}$$

For $\theta \in \mathsf{T}_{t,T}$, we have

$$\begin{aligned} Y_t &= e^{-\int_t^{\tau_t^* \wedge \theta} r_u du} Y_{\tau_t^* \wedge \theta} + \int_t^{\tau_t^* \wedge \theta} e^{-\int_t^s r_u du} dK_s^+ - \int_t^{\tau_t^* \wedge \theta} e^{-\int_t^s r_u du} dK_s^- \\ &\quad - \sum_{k=1}^{\infty} \int_t^{\tau_t^* \wedge \theta} e^{-\int_t^s r_u du} Z_s^{(k)} d\tilde{H}_s^{(k)}. \end{aligned}$$

Since $\int_t^{\tau_t^* \wedge \theta} e^{-\int_t^s r_u du} dK_s^- = 0$ and $\int_t^{\tau_t^* \wedge \theta} e^{-\int_t^s r_u du} dK_s^+ \geq 0$, then

$$\begin{aligned} Y_t &\geq \mathsf{E}_{\mathbb{Q}}\left(e^{-\int_t^{\tau_t^* \wedge \theta} r_u du} Y_{\tau_t^* \wedge \theta} / \mathcal{F}_t \right) \\ &\geq \mathsf{E}_{\mathbb{Q}}\left(e^{-\int_t^{\tau_t^*} r_u du} \mathbf{U}(\tilde{S}_{\tau_t^*})\, \mathsf{I}_{\{\tau_t^* < \theta\}} + e^{-\int_t^{\theta} r_u du} \mathbf{L}(\tilde{S}_\theta)\, \mathsf{I}_{\{\theta \leq \tau_t^*\}} + e^{-\int_t^T r_u du} g(\tilde{S}_T)\, \mathsf{I}_{\{\theta = \tau_t^* = T\}} / \mathcal{F}_t \right) \\ &= \mathsf{E}_{\mathbb{Q}}[\mathsf{J}(\tau_t^*, \theta) / \mathcal{F}_t]. \end{aligned} \tag{24}$$

Also, for $\tau \in \mathsf{T}_{t,T}$, we have

$$\begin{aligned} Y_t &= e^{-\int_t^{\tau \wedge \theta_t^*} r_u du} Y_{\tau \wedge \theta_t^*} + \int_t^{\tau \wedge \theta_t^*} e^{-\int_t^s r_u du} (dK_s^+ - dK_s^-) \\ &\quad - \sum_{k=1}^{\infty} \int_t^{\tau \wedge \theta_t^*} e^{-\int_t^s r_u du} Z_s^{(k)} d\tilde{H}_s^{(k)}. \end{aligned}$$

Since $\int_t^{\tau \wedge \theta_t^*} e^{-\int_t^s r_u du} dK_s^+ = 0$ and $-\int_t^{\tau \wedge \theta_t^*} e^{-\int_t^s r_u du} dK_s^- \leq 0$, then

$$Y_t \leq \mathsf{E}_{\mathbb{Q}}\left(e^{-\int_t^{\tau\wedge\theta_t^*} r_u du} Y_{\tau\wedge\theta_t^*}/\mathcal{F}_t \right)$$

$$\leq \mathsf{E}_{\mathbb{Q}}\left(e^{-\int_t^{\tau} r_u du}\mathbf{U}(\tilde{S}_\tau)\,\mathsf{I}_{\{\tau<\theta_t^*\}} + e^{-\int_t^{\theta_t^*} r_u du}\mathbf{L}(\tilde{S}_{\theta^*})\,\mathsf{I}_{\{\theta_t^*\leq\tau\}} \right.$$

$$\left. + e^{-\int_t^{T} r_u du} g(\tilde{S}_T)\,\mathsf{I}_{\{\theta_t^*=\tau=T\}}/\mathcal{F}_t \right) = \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta_t^*)/\mathcal{F}_t]. \quad (25)$$

In force of inequalities (23), (24) and (25) we obtain that for all $\theta,\tau \in \mathsf{T}_{t,T}$

$$\mathsf{E}_{\mathbb{Q}}[\Im(\tau_t^*,\theta)/\mathsf{F}_t] \leq Y_t = \mathsf{E}_{\mathbb{Q}}[\Im(\tau_t^*,\theta_t^*)/\mathcal{F}_t] \leq \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta_t^*)/\mathcal{F}_t].$$

Then it is immediately checked that

$$\inf_{\tau\in\mathsf{T}_{t,T}} \sup_{\theta\in\mathsf{T}_{t,T}} \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta)/\mathcal{F}_t] \leq \sup_{\theta\in\mathsf{T}_{t,T}} \mathsf{E}_{\mathbb{Q}}[\Im(\tau_t^*,\theta)/\mathcal{F}_t] \leq \mathsf{E}_{\mathbb{Q}}[\Im(\tau_t^*,\theta_t^*)/\mathcal{F}_t]$$

$$\leq \inf_{\tau\in\mathsf{T}_{t,T}} \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta_t^*)/\mathcal{F}_t] \leq \sup_{\theta\in\mathsf{T}_{t,T}} \inf_{\tau\in\mathsf{T}_{t,T}} \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta)/\mathcal{F}_t].$$

We have also that

$$\sup_{\theta\in\mathsf{T}_{t,T}} \inf_{\tau\in\mathsf{T}_{t,T}} \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta)/\mathcal{F}_t] \leq \inf_{\tau\in\mathsf{T}_{t,T}} \sup_{\theta\in\mathsf{T}_{t,T}} \mathsf{E}_{\mathbb{Q}}[\Im(\tau,\theta)/\mathcal{F}_t].$$

Theorem 5.1 is then proved. ■

6. Conclusion

In this paper, we have formulated a notion of doubly reflected BSDEs driven by a Lévy process in the case of right continuous barriers and stochastic Lipschitzian coefficient. We have also given an application to the pricing of American game options in a Lévy market.

Regarding the perspectives, it would be useful to establish, in the Markovian case, some connections between doubly reflected BSDEs driven by a Lévy process and related obstacle problems where the results provided in the present paper can be valuable tools. It is interesting to provide analogous results when the noise is driven by an inhomogeneous Lévy process. There are still some open questions related to the case when the reflecting barriers are not necessarily right continuous.

Appendix

A. **Special BSDELs**

In this section we give a special case of existence and uniqueness result of BSDELs when the coefficient depends only on y. Consider the following BSDEL:

$$Y_t = \xi + \int_t^T h(s,Y_s)ds - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k)} dH_s^{(k)}, \quad (A.1)$$

where $\xi \in \mathcal{L}^2(\beta,A,\mathbb{R})$, $\frac{h(.,0)}{a} \in \mathsf{H}^2(\beta,A,\mathbb{R})$ and there exists a positive constant κ such that $|h(t,y) - h(t,y')| \leq \kappa|y - y'|$.

Theorem A. 1. *The BSDEL (A.1) admits a unique solution* $(Y,Z) \in M^{2,c}(\beta,A,\mathbb{R})$.

Theorem A. 2. *Let* (Y^i,Z^i) *be a solution to BSDEL (A.1) associated with parameters* (ξ^i,h^i) *for* $i = 1,2$. *If* $\xi^1 \leq \xi^2$ *and* $h^1 \leq h^2$ *then* $Y^1 \leq Y^2$ *a.s.*

The Theorems A.1 and A.2 have been shown by El Jamali and El Otmani [50] in the general case when f depends on y and z, and the Lévy process is inhomogeneous.

B. **Special RBSDELs**

Here we study a special case for RBSDEL when the generator does not depend on (y,z). We consider the following RBSDEL for all $t \leq T$

$$\begin{cases} Y_t = \xi + \int_t^T h(s)ds + K_T - K_t - \sum_{k=1}^{\infty} \int_t^T Z_s^{(k)} dH_s^{(k)}, \\ Y_t \geq L_t \ \mathsf{P}-\text{a.s.} \\ \int_0^T (Y_t - L_t)dK_t^c = 0 \quad \text{and} \quad K_t^d = \sum_{0<s<t}(Y_s - L_{s-})^- \mathsf{I}_{\{\Delta L_s < 0\}}, \end{cases} \tag{B.1}$$

where $\xi \in \mathcal{L}^2(\beta,A,\mathbb{R})$, $L^+ \in \mathsf{S}^2(2\beta,A,\mathbb{R})$ and $\frac{h}{a} \in \mathsf{H}^2(\beta,A,\mathbb{R})$.

Theorem B. 1. *The RBSDEL (B.1) admits a unique solution* $(Y,Z,K) \in M^{2,c}(\beta,A,\mathbb{R}) \times K^2$ *such that*

$$\begin{aligned} &\mathsf{E} \sup_{0\leq t\leq T} e^{\beta A_t}|Y_t|^2 + \mathsf{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_s\|^2_{\ell^2} ds + \mathsf{E}|K_T|^2 \\ &\qquad \leq C\left(\mathsf{E}e^{\beta A_T}|\xi|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{h(s)}{a_s}\right|^2 ds + \mathsf{E}\sup_{0\leq t\leq T} e^{2\beta A_s}|L_s^+|^2 \right). \end{aligned} \tag{B.2}$$

Proof. For each $n \in \mathbb{N}$, we consider the following penalized version of RBSDEL (B.1)

$$Y_t^n = \xi + \int_t^T h(s)ds + n\int_t^T (Y_s^n - L_s)^- ds - \sum_{k=1}^{\infty} \int_t^T Z_s^{(k),n} dH_s^{(k)}. \tag{B.3}$$

We denote $K_t^n := n\int_0^t (Y_s^n - L_s)^- ds$ and $h^n(t,y) = h(t) + n(y - L_t)^-$. Remark that h^n is n-Lipschitz and

$$\mathsf{E}\int_0^T e^{\beta A_t}\left|\frac{h^n(t,0)}{a_t}\right|^2 dt \leq 2\mathsf{E}\int_0^T e^{\beta A_t}\left|\frac{h(t)}{a_t}\right|^2 dt + \frac{2n^2 T}{\epsilon}\mathsf{E}\sup_{0\leq t\leq T} e^{2\beta A_t}|L_t^+|^2.$$

From Theorem A.1, there exists a unique process $(Y^n,Z^n) \in \mathsf{M}^{2,c}(\beta,A,\mathbb{R})$ solution of the BSDEL $(B.3)$. Due to the convergence results in El Jamali and El Otmani [53], we have

$$\begin{cases} Y^n \to Y \text{ in } \mathsf{B}^2(\beta,A,\mathbb{R}) \text{ as } n \text{ tends to } +\infty, \\ Z^n \to Z \text{ in } \mathsf{H}^2(\beta,A,\ell^2) \text{ as } n \text{ tends to } +\infty, \\ K^n \to K \text{ in } \mathsf{K}^2 \text{ as } n \text{ tends to } +\infty. \end{cases}$$

Furthermore, the process (Y,Z,K) is the unique solution of the RBSDEL $(B.1)$ associated with parameters (ξ,h,L) such that

$$\mathsf{E}\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2 + \mathsf{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_s\|_{\ell^2}^2 ds + \mathsf{E}|K_T|^2$$
$$\le C\left(\mathsf{E}e^{\beta A_T}|\xi|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{h(s)}{a_s}\right|^2 ds + \mathsf{E}\sup_{0\le t\le T} e^{2\beta A_s}|L_s^+|^2\right).$$

Here the proof ends.

■

Corollary B. 1. *For all $t \le T$, the RBSDEL with upper barrier $(U_t)_{t\le T}$ define as*

$$\begin{cases} Y_t = \xi + \int_t^T h(s)ds - K_T + K_t - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k)} dH_s^{(k)}, \\ Y_t \le U_t, \quad \int_0^T (U_t - Y_t)dK_t^c = 0 \quad \text{and} \quad K_t^d = \sum_{0<s<t}(Y_s - U_{s-})^+ \mathsf{I}_{\{\Delta U_s>0\}}, \end{cases}$$

admits a unique solution $(Y,Z,K) \in M^{2,c}(\beta,A,R) \times K^2$ such that

$$\mathsf{E}\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2 + \mathsf{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \mathsf{E}\int_0^T e^{\beta A_s}\|Z_s\|_{\ell^2}^2 ds + \mathsf{E}|K_T|^2$$
$$\le C\left(\mathsf{E}e^{\beta A_T}|\xi|^2 + \mathsf{E}\int_0^T e^{\beta A_s}\left|\frac{h(s)}{a_s}\right|^2 ds + \mathsf{E}\sup_{0\le t\le T} e^{2\beta A_s}|U_s^-|^2\right). \quad (B.4)$$

Theorem B. 2. *Let (Y^i,Z^i,K^i) be a solution to RBSDEL associated with parameters (ξ^i,h^i,L) of the form*

$$\begin{cases} Y_t^i = \xi + \int_t^T h(s,Y_s^i)ds + K_T^i - K_t^i - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k),i} dH_s^{(k)}, \\ Y_t^i \ge L_t, \quad \int_0^T (Y_t^i - L_t)dK_t^{i,c} = 0 \quad \text{and} \quad K_t^{i,d} = \sum_{0<s<t}(Y_s^i - L_{s-})^- \mathsf{I}_{\{\Delta L_s<0\}} \end{cases} \quad (B.5)$$

for $i = 1,2$ and $t \le T$. If $h^1(t,y) \le h^2(t,y)$ a.s. and $\xi^1 \le \xi^2$ a.s. Then $Y_t^1 \le Y_t^2$ and $K_t^1 \ge K_t^2$ $\forall t \le T$ a.s.

Proof. We consider the penalized equations relative to the RBSDEL $(B.5)$, for $i = 1,2$ and $n \in \mathsf{N}$, as follows

$$Y_t^{n,i} = \xi^i + \int_t^T h^i(s,Y_s^{n,i})ds + n\int_t^T (Y_s^{n,i} - L_s)^- - \sum_{k=1}^{\infty}\int_t^T Z_s^{(k),n,i} dH_s^{(k)}.$$

By Theorem A.2, on one hand, we have $Y^{n,1} \leq Y^{n,2}$, for any $n \in \mathbb{N}$. On the other hand, it has been shown by El Jamali and El Otmani [53] that the sequence $(Y_t^{n,i})_{n\geq 0}$ converges increasingly to Y_t^i a.s. Therefore, at least after extracting a subsequence, the sequence $\left(n\int_0^t (Y_s^{n,i} - L_s)^- ds\right)_{n\geq 0}$ converge to K_t^i for $i = 1,2$. Henceforth, for any $0 \leq t \leq T$ we have

$$Y_t^1 \leq Y_t^2 \quad \text{and} \quad K_t^1 = \lim_{n\to+\infty} n\int_0^t (Y_s^{n,1} - L_s)^- ds$$
$$\geq \lim_{n\to+\infty} n\int_0^t (Y_s^{n,2} - L_s)^- ds = K_t^2.$$

Since L is a rcll process then K is also rcll process which includes the purely jumping part K^d where

$$K_t^{1,d} = \lim_{n\to+\infty} n\sum_{0<s<t} (Y_s^{n,1} - L_{s-})^- 1_{\{\Delta L<0\}}$$
$$\geq \lim_{n\to+\infty} n\sum_{0<s<t} (Y_s^{n,2} - L_{s-})^- 1_{\{\Delta L<0\}} = K_t^{2,d}.$$

Here the proof ends. ■

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